C*-ALGEBRAIC CHARACTERIZATION OF BOUNDED ORBIT INJECTION EQUIVALENCE FOR MINIMAL FREE CANTOR SYSTEMS

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ABSTRACT. Bounded orbit injection equivalence is an equivalence relation defined on minimal free Cantor systems which is a candidate to generalize flip Kakutani equivalence to actions of the Abelian free groups on more than one generator. This paper characterizes bounded orbit injection equivalence in terms of a mild strengthening of Rieffel-Morita equivalence of the associated C*-crossed-product algebras. Moreover, we construct an ordered group which is an invariant for bounded orbit injection equivalence, and does not agrees with the K_0 group of the associated C*-crossed-product in general. This new invariant allows us to find sufficient conditions to strengthen bounded orbit injection equivalence to orbit equivalence and strong orbit equivalence.

1. INTRODUCTION

This paper establishes a characterization of bounded orbit injection equivalence, as introduced in [7] by S. Lightwood and the second author, in terms of a strengthened form of Rieffel-Morita equivalence between C*-crossed-products. For minimal $\mathbb{Z}$-actions of the Cantor set, bounded orbit injection equivalence is equivalent to flip-Kakutani equivalence, i.e., the equivalence relation generated by Kakutani equivalence and time reversal. Bounded orbit injection equivalence is a generalization of flip-Kakutani equivalence which applies to actions of $\mathbb{Z}^d$ where time reversal is not a well-defined concept. While Giordano, Putnam and Skau have shown in [12, Theorem 2.6] that Kakutani strong orbit equivalence is characterized by Rieffel-Morita equivalence of the C*-crossed-products, the C*-algebraic picture of Kakutani equivalence, and more generally bounded orbit injection equivalence, is the main new result of this article and, informally, can be described as a form of Rieffel-Morita equivalence where moreover the space on which the action occurs is remembered. We begin our paper with an introduction of the concepts we will use and the framework for our characterization. We then establish our characterization in the next section. In the last section we apply our results to derive a sufficient condition for bounded orbit injections to give rise to strong orbit equivalence. This condition involves an ordered group which is a direct $\mathbb{Z}^d$ analog of the group used in [12, Theorem 2.6] when $d = 1$, but in contrast to that case, it is not the K_0-group of the C*-algebra when $d > 1$. Giordano, Matui, Putnam and Skau have recently

shown that this group modulo the infinitesmal subgroup characterizes minimal $\mathbb{Z}^d$ Cantor systems up to orbit equivalence [14].

A triple $(X, \varphi, \mathbb{Z}^d)$ is a dynamical system (on a compact space) when X is a compact space and φ is an action of $\mathbb{Z}^d$ on X by homeomorphisms. A Cantor system $(X, \varphi, \mathbb{Z}^d)$ is a dynamical system where X is a Cantor set. Moreover, if φ is a free action, then $(X, \varphi, \mathbb{Z}^d)$ will be called free as well, and if every point in X has a dense orbit for the action φ then $(X, \varphi, \mathbb{Z}^d)$ will be called minimal.

Let $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ be two free minimal Cantor systems for some positive integer d. We wish to investigate when φ and ψ are equivalent in some dynamically meaningful way. A natural concept of equivalence is given by conjugacy: φ and ψ are conjugate when there exists a homeomorphism $h : X \to Y$ such that $h \circ \varphi = \psi \circ h$. However, classification up to conjugacy is a very complex problem, and it has proven fruitful to study weaker form of equivalences with more tractable invariants. A fundamental example of such an equivalence is orbit equivalence [12]: the actions φ and ψ of $\mathbb{Z}$ are orbit equivalent when there exists a homeomorphism $h : X \to Y$ and two maps $n : X \to \mathbb{Z}$ and $m : Y \to \mathbb{Z}$ such that for all $x \in X$ and $y \in Y$ we have $h \circ \varphi(x) = \psi^{n(x)} \circ h(x)$ and $\varphi^{m(y)} \circ h^{-1}(y) = h^{-1} \circ \psi(y)$. In other words, φ and ψ are orbit equivalent if and only if there exists a homeomorphism $h : X \to Y$ which maps orbits to orbits. Following [12], the actions φ and ψ are strongly orbit equivalent when the maps m and n are discontinuous at one point at most. Giordano, Putnam and Skau proved in [12, Theorem 2.6] that the C*-crossed-product algebra of the two minimal free actions φ and ψ of $\mathbb{Z}$ on a Cantor set are *-isomorphic if and only if φ and ψ are strongly orbit equivalent.

The C*-algebra of an action α of $\mathbb{Z}^d$ on a compact set X is defined as the universal C*-algebra $C(X) \rtimes_\alpha \mathbb{Z}^d$ generated by $C(X)$ and unitary operators U_α^z for all $z \in \mathbb{Z}^d$ subject to the relations $U_\alpha^z f (U_\alpha^z)^* = f \circ \alpha^{-z}$ and $U_\alpha^z U_\alpha^{z'} = U_\alpha^{z+z'}$ for all $f \in C(X)$ and $z, z' \in \mathbb{Z}^d$ with $U_\alpha^0 = 1$. C*-crossed-products were introduced in Zeller-Meier in [17] and have a rich and complex structure as C*-algebras [17][9] [16], whose connection with the defining action is not always clear. It is thus a remarkable fact that when α is a minimal and free action of $\mathbb{Z}$ on a Cantor set X, the crossed-product $C(X) \rtimes_\alpha \mathbb{Z}$ is a complete invariant for strong orbit equivalence. Moreover, in this case, the C*-crossed-products are inductive limits of so called circle algebras and are fully classified up to *-isomorphism by their K groups (including the order on K_0 and some distinguished elements in each group K_0 and K_1) , as shown by Elliott in [3]. Thus, the ordered K_0 group and its order unit of the C*-crossed-product for such actions form a complete invariant of strong orbit equivalence.

Yet, in general, proving that two C*-algebras are *-isomorphic is nontrivial, and again some weaker but interesting form of equivalence have been introduced to help with this problem. Two C*-algebras A and B are Rieffel-Morita equivalent when, informally, their categories of non-degenerate representations on Hilbert spaces are equivalent. More formally, two C*-algebras A and B are Rieffel-Morita equivalent when there exists a full B Hilbert module M such that the C*-algebra of compact adjoinable operators on M is *-isomorphic to A [6, Theorem 4.26 p. 164][15]. In particular, when the C*-algebras A and B are simple, if there exists a *-isomorphism $\varphi : A \to pBp$ with p some nonzero projection in B then A and B are Morita equivalent. In [12, Theorem 2.6], Giordano, Putnam and Skau established that

$C(X) \rtimes_\varphi \mathbb{Z}$ and $C(Y) \rtimes_\psi \mathbb{Z}$ are Rieffel-Morita equivalent if and only if φ and ψ are Kakutani strong orbit equivalent. Kakutani strong orbit equivalence is defined in terms of induced systems: if A is a clopen subset of a Cantor set Z and α is a minimal free action of $\mathbb{Z}$ on Z then α^A is the action of $\mathbb{Z}$ on A defined by first return times of α to A, and is called a derived system of α. Now, φ and ψ are strongly orbit Kakutani equivalent if there exists a free minimal action α of $\mathbb{Z}$ on some Cantor set Z such that both φ and ψ are conjugated to derived systems of α. The systems φ and ψ are Kakutani strong orbit equivalent when they are strongly orbit equivalent to derived systems of α. Thus once again, a dynamical concept such as Kakutani strong orbit equivalence is characterized by a standard concept of C*-algebra theory — Rieffel-Morita equivalence. It is thus natural to investigate analogous C*-algebraic descriptions of other forms of equivalence between minimal free Cantor systems.

Two minimal free systems $(X, \varphi, \mathbb{Z})$ and $(Y, \psi, \mathbb{Z})$ are flip-Kakutani equivalent if $(X, \varphi, \mathbb{Z})$ is Kakutani equivalent to either $(Y, \varphi, \mathbb{Z})$ or its time reversed system $(Y, \varphi \circ \sigma, \mathbb{Z})$ where $\sigma : z \in \mathbb{Z} \mapsto -z$. Now, σ is the only automorphism of $\mathbb{Z}$ besides the identity, but the situation is more complicated for $\mathbb{Z}^d$ in general. To generalize the notion of flip-Kakutani equivalence, S. Lightwood and the second author introduced in [7] the notion of orbit injection equivalence between minimal free actions of $\mathbb{Z}^d$ on the Cantor set. In general, for an action α of $\mathbb{Z}^d$ on a set Z we denote the image of $z \in \mathbb{Z}^d$ by α^z. We recall from [7]:

Definition 1.1. *Let* $(X, \varphi, \mathbb{Z}^d)$ *and* $(Y, \psi, \mathbb{Z}^d)$ *be two free dynamical systems. Then a map* $\theta : X \to Y$ *is an* orbit injection *from* $(X, \varphi, \mathbb{Z}^d)$ *to* $(Y, \psi, \mathbb{Z}^d)$ *when* θ *is a continuous open injection such that for all* $x, y \in X$ *we have:*
(1.1)
$$\exists w \in \mathbb{Z}^d \text{ such that } \varphi^w(x) = y \text{ if and only if } \exists v \in \mathbb{Z}^d \text{ such that } \psi^v \circ \theta(x) = \theta(y).$$

The cocycle *for an orbit injection* θ *is the function* $\eta : X \times \mathbb{Z}^d \to \mathbb{Z}^d$ *defined by* $\psi^{\eta(x,w)} \theta(x) = \theta(\varphi^w x)$. *The orbit injection* θ *is called* bounded *if the cocycle* η *is continuous.*

We observe that, as the actions φ and ψ are free, Identity (1.1) uniquely defines the cocycle η. Moreover, the range of an orbit injection is clopen, since it is assumed open by definition and it is the continuous image of a compact set so it is closed as well.

Remark 1.2. *In* [7], *bounded orbit injections are not required to have open range, and it is shown instead that for* $d = 2$ *the range of bounded orbit injections has nonempty interior. In general, given two minimal free Cantor systems* $(X, \varphi, \mathbb{Z}^d)$ *and* $(Y, \psi, \mathbb{Z}^d)$, *the range of a bounded orbit injection* θ *from* $(X, \varphi, \mathbb{Z}^d)$ *to* $(Y, \psi, \mathbb{Z}^d)$ *has nonempty interior if and only if it is open (or equivalently* θ *is an open map). Indeed, assume* $X' = \theta(X)$ *contains an open subset* U. *Let* $y \in X'$. *By minimality there exists* $z \in \mathbb{Z}^d$ *and* $\omega \in U$ *such that* $\psi^z(\omega) = y$. *Then there exists a unique* $x \in X$ *and* $h \in \mathbb{Z}^d$ *such that* $\theta(x) = \omega$ *and* $\theta(\varphi^{-h}(x)) = y$ *since* θ *is an orbit injection. Now, since the cocycle* η *associated to* θ *is continuous, the subset* $\Omega = (\eta(\cdot, z))^{-1}(\{-h\})$ *of* X *is open. Let* $V = \Omega \cap \theta^{-1}(U)$ *which is open in* X *as well. Now, since* X *is compact and* θ *is a continuous injection,* θ *is a homeomorphism from* X *onto* X' *for its relative topology in* Y. *Thus in particular,* $\theta(V)$ *is relatively*

open in X', i.e. there exists an open subset Υ of Y such that $\theta(V) = \Upsilon \cap X'$. Yet $\theta(V) \subseteq U \subseteq X'$ and U is open in Y so $\theta(V) = \Upsilon \cap U$ is open in Y. Moreover, by construction, $\psi^z(\theta(V)) = \theta(\varphi^{-h}(V))$ so $\psi^z(\theta(V))$ is an open subset of Y (since ψ^z is a homeomorphism) and is contained in $\theta(X)$. Yet $y \in \psi^z(\theta(V))$, so $\theta(X)$ is a neighborhood of each of its points and is thus open in Y. The converse is immediate.

Definition 1.3. *Let $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ be two free minimal Cantor systems. Then $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ are* bounded orbit injection equivalent *when there exists a minimal free Cantor system $(Z, \alpha, \mathbb{Z}^d)$ with two bounded orbit injections θ_φ and θ_ψ from, respectively, $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ to $(Z, \alpha, \mathbb{Z}^d)$.*

The fact that bounded orbit injection equivalence is reflexive and symmetric is obvious, and transitivity was proven indirectly in [7] for $d = 2$ where bounded orbit injection equivalence is proven to be the same relation as suspension equivalence. In the last section of this paper, we will give a proof that bounded orbit injection equivalence is transitive for any $d \in \mathbb{N} \setminus \{0\}$ and thus is indeed an equivalence relation.

This paper establishes that the systems $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ are bounded orbit injection equivalent if and only if both $C(X) \rtimes_\varphi \mathbb{Z}^d$ and $C(Y) \rtimes_\psi \mathbb{Z}^d$ can be embedded as corners in $C(Z) \rtimes_\alpha \mathbb{Z}^d$ for some minimal free Cantor system $(Z, \alpha, \mathbb{Z}^d)$, and the images by these embeddings of both $C(X)$ and $C(Y)$ are subalgebras of $C(Z)$. We note that this result is new even in the case of actions of $\mathbb{Z}$ as a characterization of flip-Kakutani of minimal free Cantor systems. The proof of this first result occupies our first section. It partly relies upon techniques inspired by [12, Theorem 2.6], as well as other tools such as spectral decomposition of C*-crossed-products.

We show in Theorem 3.12 that for a minimal Cantor system $(X, \varphi, \mathbb{Z}^d)$ the ordered group $(G(\varphi), G(\varphi)_+)$ where

$$
\begin{aligned}
G(\varphi) &= C(X, \mathbb{Z}) / \langle f - f\varphi^v : v \in \mathbb{Z}^d \rangle \\
G(\varphi)_+ &= \{[f] : f(x) \geq 0 \text{ for all } x \in X\}
\end{aligned}
$$

is an invariant of bounded orbit injection equivalence. In the case when φ is a $\mathbb{Z}$-action, by the Pimsner-Voiculescu six term exact sequence [1] this is the K_0-group of the C*-algebra $C(X) \rtimes_\varphi \mathbb{Z}$. We show that $(G(\varphi), G(\varphi)_+)$ shares many properties with simple dimension groups. However, as is shown in [8], this group may contain torsion even when $d = 2$. The connection between the ordered group $(G(\varphi), G(\varphi)_+)$ and the C*-algebra $C(X) \rtimes_\varphi \mathbb{Z}^d$ for $d > 1$ is unclear (see discussion in [11][10]). Thus while it is not particularly surprising that it is an invariant of bounded orbit injection equivalence, it is interesting that by the aforementioned results it is an invariant of this strengthened notion of Reiffel-Morita equivalence for $C(X) \rtimes_\varphi \mathbb{Z}^d$.

We show that given bounded orbit injections θ_1, θ_2 from two minimal Cantor systems $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ into a third $(Z, \alpha, \mathbb{Z}^d)$, the question of whether the system $(Z, \alpha, \mathbb{Z}^d)$ may be omitted turns on the nature of the isomorphisms h_{θ_1}

and h_{θ_2} from $\left(G\left(\varphi\right), G\left(\varphi\right)_+\right)$ and $\left(G\left(\psi\right), G\left(\psi\right)_+\right)$ into $\left(G\left(\alpha\right), G\left(\alpha\right)_+\right)$ induced by the orbit injections, or if one prefers, the nature of the isomorphism $h = h_{\theta_2}^{-1} h_{\theta_1} :$ $\left(G\left(\varphi\right), G\left(\varphi\right)_+\right) \rightarrow \left(G\left(\psi\right), G\left(\psi\right)_+\right)$. Specifically, letting $[1_X]$, $[1_Y]$ represent the equivalence classes of the constant functions 1 on the spaces X, Y, if $h[1_X] = [1_Y]$ then we show that there is a bounded orbit equivalence from $\left(X, \varphi, \mathbb{Z}^d\right)$ to $\left(Y, \psi, \mathbb{Z}^d\right)$. If $h[1_X] - [1_Y]$ is an infinitesimal element then we show that the orbit injections can be modified to a (not necessarily bounded) orbit equivalence from $\left(X, \varphi, \mathbb{Z}^d\right)$ to $\left(Y, \psi, \mathbb{Z}^d\right)$. Finally, if $[1_Y] - h[1_X] \in G\left(\psi\right)_+$ then there is a bounded orbit injection from X to Y.

As is the case in [7, 14], partitions associated with tilings are a key tool for proving results about $\mathbb{Z}^d$-dynamics and we use Voronoi tilings for our results about the group $\left(G\left(\varphi\right), G\left(\varphi\right)_+\right)$. In particular, the notion of a Voronoi-Rohlin partition serves as a $\mathbb{Z}^d$ version of the tower partitions used in [12, Theorem 2.6].

2. C*-algebraic Characterization of Orbit Injection Equivalence

This section establishes the characterization of bounded orbit injection equivalence in terms of Rieffel-Morita equivalence of C*-algebras. Given a dynamical system $\left(X, \varphi, \mathbb{Z}^d\right)$ on a compact space X, the C*-crossed-product $C(X) \rtimes_\varphi \mathbb{Z}^d$ is the universal C*-algebra generated by a copy of $C(X)$ and a family $\left(U_\varphi^z\right)_{z \in \mathbb{Z}^d}$ of unitary operators satisfying the relations $U_\varphi^z f U_\varphi^{-z} = f \circ \varphi^{-z}$, $U_\varphi^z U_\varphi^{z'} = U_\varphi^{z+z'}$ and $U_\varphi^0 = 1$ for all $z, z' \in \mathbb{Z}^d$ and $f \in C(X)$. The construction of this C*-algebra is detailed in [17],[9] and [16]. Note that in this paper, we shall follow the convention that φ^z is the homeomorphism given by the action φ on X at $z \in \mathbb{Z}^d$, and the associated unitary in the crossed-product $C(X) \rtimes_\varphi \mathbb{Z}^d$, which we will call the canonical unitary for φ at z, is denoted by U_φ^z.

This section generalizes methods developed in [12] to the case of $\mathbb{Z}^d$ actions. An important tool in [12] is the description of normalizers of the C*-subalgebra $C(X)$ in the crossed-product $C(X) \rtimes \mathbb{Z}$. We shall also need such a description, but in addition we need to make sure that the normalizers we will encounter form a group isomorphic to $\mathbb{Z}^d$ — something which is automatic when $d = 1$ but needs some efforts for the general case we study in this paper. In general, the generalization from actions of $\mathbb{Z}$ to actions of $\mathbb{Z}^d$ involve various technical points which we emphasize in the proofs of this section.

As a first step, we shall establish the following proposition, which provides a mean to embed a crossed-product into another given a bounded orbit injection:

Proposition 2.1. *Let X and Z be Cantor sets. Let $\left(X, \varphi, \mathbb{Z}^d\right)$ and $\left(Z, \psi, \mathbb{Z}^d\right)$ be two free dynamical systems and $\theta : X \rightarrow Z$ a bounded orbit injection with unique cocycle η. Then the projections defined for $z, h \in \mathbb{Z}^d$ and $y \in Z$ by:*

(2.1) $p_h^z(y) = \begin{cases} 1 \text{ if } \exists x, x' \in X \quad y = \theta(x) \text{ and } x = \varphi^z(x') \text{ and } y = \psi^h\left(\theta\left(x'\right)\right), \\ \qquad\qquad 0 \text{ otherwise.} \end{cases}$

satisfy:

- *For $z, h \in \mathbb{Z}^d$, the projection p_h^z is in $C(Z)$, hence in $C(Z) \rtimes_\psi \mathbb{Z}^d$,*
- *For $z \in \mathbb{Z}^d$ and $h \neq h' \in \mathbb{Z}^d$ we have $p_h^z p_{h'}^z = 0$,*
- *For $z \in \mathbb{Z}^d$ the set $\left\{ h \in \mathbb{Z}^d : p_h^z \neq 0 \right\}$ is finite,*
- *The set $\theta(X)$ is clopen in Z,*
- *If p is the projection on $\theta(X)$ i.e. is the multiplication operator by the indicator of $\theta(X)$ in Z, then for all $z \in \mathbb{Z}^d$ we have:*

$$\sum_{h \in \mathbb{Z}^d} p_h^z = p,$$

- *For $z, z', h \in \mathbb{Z}^d$ we have:*

$$p_h^{z+z'} = \sum_{h' \in \mathbb{Z}^d} p_{h'}^z \cdot p_{h-h'}^{z'} \circ \psi^{-h'},$$

Consequently, setting for $z \in \mathbb{Z}^d$:

$$V^z = \sum_{h \in \mathbb{Z}^d} p_h^z U_\psi^h + (1 - p)$$

then the map $z \in \mathbb{Z}^d \mapsto V^z$ is a group isomorphism into the unitary group in $C(Z) \rtimes_\psi \mathbb{Z}^d$. Moreover, $V^z p = p V^z$ for all $z \in \mathbb{Z}^d$.

Proof. We denote $\theta(X)$ by X'. By assumption, θ is a continuous bijection from X onto X'. Since X is compact, θ is in fact a homeomorphism from X onto X' for the relative topology on X'. Moreover, as the continuous image of a compact set, X' is closed in Z. Since by assumption on θ the set X' is also open in Z, we conclude that $\theta(X)$ is clopen in Z.

For all $y \in Z$ and $z, h \in \mathbb{Z}^d$, we define $p_h^z(y)$ by Identity (2.1) and we note that the support X_h^z of p_h^z is the image by θ of set $P_h^z = (\eta(\cdot, z))^{-1}(\{h\})$, the latter being clopen in X since η is continuous. Since X' is clopen in Z and θ is a homeomorphism from X onto X' we deduce that X_h^z is clopen in Z. Hence p_h^z is a continuous function on Z. We now establish the properties of the lemma.

Assume $X_h^z \cap X_{h'}^z \neq \emptyset$ for some $z, h, h' \in \mathbb{Z}^d$. Let $y \in X_h^z \cap X_{h'}^z$. Then by definition, there exists $x, x', x'', x''' \in X$ such that $y = \theta(x) = \theta(x'')$, with $x = \varphi^z(x')$ and $x'' = \varphi^z(x''')$ and $y = \psi^h(\theta(x')) = \psi^{h'}(\theta(x'''))$. Since θ is injective, $x = x''$. Since φ^z is a homeomorphism, we have $x' = x'''$. Since the action by ψ is free, $h = h'$. Thus, if $h \neq h'$ then $X_h^z \cap X_{h'}^z = \emptyset$.

Moreover, let $y \in X'$. By definition, there exists (a unique) $x \in X$ such that $y = \theta(x)$. Set $x' = \varphi^{-z}(x)$. Since θ is an orbit injection with cocycle η, we have:

$$\psi^{\eta(x,z)}(\theta(x')) = \theta(\varphi^z(x')) = \theta(x) = y$$

by Definition (1.1). Hence by definition $y \in X_{\eta(x,z)}^z$. Conversely, if $y \in X_h^z$ then by definition $y \in X'$. Thus for any $z \in \mathbb{Z}^d$ the set X' is the union of the clopen sets X_h^z for $h \in \mathbb{Z}^d$ and thus X' itself is clopen. Furthermore, as X' is closed in the compact set X we conclude that X' is compact. Thus for any $z \in \mathbb{Z}^d$, if $\mathcal{P}_z = \left\{ h \in \mathbb{Z}^d : X_h^z \neq \emptyset \right\}$ then the family $(X_h^z)_{h \in \mathcal{P}_z}$ is a partition of the compact X' by open subsets, so it is finite. Hence $\mathcal{P}_z$, which equals $\left\{ h \in \mathbb{Z}^d : p_h^z \neq 0 \right\}$ by definition, is finite as claimed.

We thus have proven that for a fixed $z \in \mathbb{Z}^d$, the projections p_h^z ($h \in \mathbb{Z}^d$) are pairwise orthogonal and that $\sum_{h \in \mathbb{Z}^d} p_h^z = p$ where p is the projection on X' i.e. the indicator function of X' in $C(Z)$ (note that $p \in C(Z)$ since X' is clopen in

Z) – and where only finitely many terms in the sum are nonzero. We now prove a natural convolution product relates the projections p_h^z for varying $z \in \mathbb{Z}^d$.

Let $z, z', h \in \mathbb{Z}^d$. Let $y \in Z$. Then $p_h^{z+z'}(y) = 1$ if and only if there exists $x, x' \in X$ such that:

$$(2.2) \qquad y = \theta(x), \ x = \varphi^{z+z'}(x') \text{ and } y = \psi^h(\theta(x')).$$

Since $x = \varphi^z\left(\varphi^{z'}(x')\right)$, by assumption on θ, there exists $h' = \eta\left(\varphi^{z'}(x'), z\right) \in \mathbb{Z}^d$ such that $y = \psi^{h'}\left(\theta\left(\varphi^{z'}(x')\right)\right)$. In summary, for some $h' \in \mathbb{Z}^d$:

$$(2.3) \qquad y = \psi^{h'}\left(\theta\left(\varphi^{z'}(x')\right)\right) \text{ and } x = \varphi^z\left(\varphi^{z'}(x')\right).$$

Hence by definition, $p_{h'}^z(y) = 1$. Note moreover that h' is unique, since $p_{h'}^z(y) = 1$ implies that $p_{h''}^z(y) = 0$ for all $h'' \neq h'$.

Furthermore, since $\psi^{-h'} = \left(\psi^{h'}\right)^{-1}$ we have from the first equality in (2.3):

$$\psi^{-h'}(y) = \theta\left(\varphi^{z'}(x')\right)$$

and from the last equality in (2.2):

$$\psi^{-h'}(y) = \psi^{h-h'}(\theta(x'))$$

and therefore by definition:

$$p_{h-h'}^{z'}\left(\psi^{-h'}(y)\right) = 1.$$

Hence we obtain the desired formula:

$$(2.4) \qquad p_h^{z+z'}(y) = \sum_{h' \in \mathbb{Z}^d} p_{h'}^z(y)\left(p_{h-h'}^{z'} \circ \psi^{-h'}(y)\right),$$

noting that for any $y \in X$ only one term at most in the sum is nonzero.

We can now define for all $z \in \mathbb{Z}^d$ the following operator:

$$V^z = \sum_{h \in \mathbb{Z}^d} p_h^z U_\psi^h + (1-p) \in C(Z) \rtimes_\psi \mathbb{Z}^d,$$

where again the sum is over only finitely many nonzero terms.

Now, let $y \in Z \setminus X'$ and let $z, h \in \mathbb{Z}^d$. We compute:

$$\begin{aligned}
\left(p_h^z U_\psi^h (1-p)\right)^* \left(p_h^z U_\psi^h (1-p)\right) &= (1-p)\left(p_h^z \circ \psi^h\right)(1-p) \\
&= (1-p)\left(p_h^z \circ \psi^h\right).
\end{aligned}$$

Yet if $p_h^z \circ \psi^h(y) = 1$ then by definition there exists $x, x' \in X'$ such that $\psi^h(y) = \theta(x)$, $x = \varphi^z(x')$ and $\psi^{-h}\left(\psi^h(y)\right) = \theta(x') \in X'$. This last equation forces $y \in X'$ and thus $(1-p)(y) = 0$. Hence:

$$(2.5) \qquad p_h^z U_\psi^h (1-p) = 0.$$

By the convolution formula (2.4), Equality (2.5) and the trivial observation:

$$((1-p)\, p_h^z)(y) = 0$$

for all $z, h \in \mathbb{Z}^d$ and $y \in Z$, we conclude:

$$
\begin{aligned}
V^z V^{z'} &= \sum_{h \in \mathbb{Z}^d} \sum_{h' \in \mathbb{Z}^d} p_h^z U_\psi^h p_{h'}^{z'} U_\psi^{h'} + (1 - p) \text{ yet } U_\psi^{h'} = U_\psi^{-h} U_\psi^{h+h'} \text{ so:} \\
V^z V^{z'} &= \sum_{h \in \mathbb{Z}^d} \sum_{h' \in \mathbb{Z}^d} \left(p_h^z p_{h'}^{z'} \circ \psi^{-h} \right) U_\psi^{h+h'} + (1 - p) \\
&= \sum_{h'' \in \mathbb{Z}^d} \left(\sum_{h \in \mathbb{Z}^d} p_h^z p_{h''-h}^{z'} \circ \psi^{-h} \right) U_\psi^{h''} + (1 - p) \text{ where } h'' = h + h' \\
&= \sum_{h'' \in \mathbb{Z}^d} p_{h''}^{z+z'} U_\psi^{h''} + (1 - p) = V^{z+z'}.
\end{aligned}
$$

Moreover by construction $V^0 = p + 1 - p = 1$. Hence $z \mapsto V^z$ is a morphism from $\mathbb{Z}^d$ into the unitary group of $C(Z) \rtimes_\psi \mathbb{Z}^d$. Last, we note that Identity (2.5) shows that $p_h^z U_\psi^h = p_h^z U_\psi^h p$ for all $z, h \in \mathbb{Z}^d$, which in turn establishes that for all $z \in \mathbb{Z}^d$ the unitary V^z commutes with p. Our proposition is thus proven. $\square$

As a first use of Proposition (2.1), we prove that the existence of a bounded orbit injection implies Rieffel-Morita equivalence of crossed-products with an additional property, thus establishing the necessary condition of our characterization of bounded orbit injection equivalence:

Proposition 2.2. *Let $(X, \varphi, \mathbb{Z}^d)$ and $(Z, \psi, \mathbb{Z}^d)$ be two minimal free Cantor systems and let $\theta : X \to Z$ be a bounded orbit injection from $(X, \varphi, \mathbb{Z}^d)$ to $(Z, \psi, \mathbb{Z}^d)$. Then there exists a *-monomorphism $\alpha : C(X) \rtimes_\varphi \mathbb{Z}^d \longrightarrow C(Z) \rtimes_\psi \mathbb{Z}^d$ such that $\alpha(C(X)) \subseteq C(Z)$ and whose range is the corner algebra $p \left(C(Z) \rtimes_\psi \mathbb{Z}^d \right) p$ where $p = \alpha(1)$.*

Proof. Let $f \in C(X)$ and $y \in Z$. We set:

$$
\pi(f)(y) = \begin{cases} f(x) \text{ if } y = \theta(x), \\ 0 \text{ otherwise.} \end{cases}
$$

First, $\pi(f)$ is a well-defined map from Z to $\mathbb{C}$ since θ is injective. Moreover, it is straightforward to check that $\pi(f)$ is continuous over Z since the range of θ is clopen by Proposition (2.1). Let V, p and p_h^z $(z, h \in \mathbb{Z}^d)$ be given by by Lemma (2.1), which applies since θ is a continuous orbit injection and both φ and ψ are free by assumption. Note that p is the indicator function of $\theta(X)$ in Z, so $p = \pi(1)$. Let now $f \in C(X)$ be given. We wish to check that (π, V) is a covariant representation of $(X, \varphi, \mathbb{Z}^d)$ so we compute:

$$
\begin{aligned}
V^z \pi(f) V^{-z} &= \left(\sum_{h \in \mathbb{Z}^d} p_h^z U_\psi^h + (1 - p) \right) \pi(f) \left(\sum_{h' \in \mathbb{Z}^d} p_{h'}^z U_\psi^{h'} + (1 - p) \right)^* \\
&= \sum_{h \in \mathbb{Z}^d} \sum_{h' \in \mathbb{Z}^d} p_h^z U_\psi^h \pi(f) U_\psi^{-h'} p_{h'}^z,
\end{aligned}
$$

since $(1 - p)\, \pi(f) = \pi(f)\,(1 - p) = 0$. Now, for $h, h' \in \mathbb{Z}^d$ we have:

$$
\begin{aligned}
p_h^z U_\psi^h \pi(f) U_\psi^{-h'} p_{h'}^z & = p_h^z \left(\pi(f) \circ \psi^{-h} \right) U_\psi^{h-h'} p_{h'}^z \\
& = \left(\pi(f) \circ \psi^{-h} \right) \left(p_h^z \left(p_{h'}^z \circ \psi^{h'-h} \right) \right) U_\psi^{h-h'} \\
(2.6) \qquad\qquad & = \begin{cases} 0 \ \text{if } h \neq h' \\ \pi(f) \circ \psi^{-h} \ \text{if } h = h'. \end{cases}
\end{aligned}
$$

Indeed, suppose that $p_h^z(y) = 1$ for some $h \in \mathbb{Z}^d$ and $y \in Z$. Then by definition, there exists $x, x' \in X$ such that $y = \theta(x) = \psi^h\left(\theta(x')\right)$ and $x = \varphi^z(x')$. Then:

$$
\psi^{h'-h}(y) = \psi^{h'-h}\left(\psi^h\left(\theta(x')\right)\right) = \psi^{h'}\left(\theta(x')\right).
$$

Yet if $p_{h'}^z(\psi^{h'-h}(y)) = 1$ for some $h' \in \mathbb{Z}^d$ then there exists $w, w' \in X$ such that $\psi^{h'-h}(y) = \theta(w) = \psi^{h'}(\theta(w'))$ and $w = \varphi^z(w')$. Yet then since $\psi^{h'} \circ \theta$ is injective, we conclude that $w' = x'$ and thus $x = w$ so $\psi^{h'-h}(y) = y$. Hence as ψ is free we conclude that $h = h'$. Thus $p_h^z \left(p_{h'}^z \circ \psi^{h'-h} \right)(y)$ is 1 when $h = h'$ and 0 otherwise.

Hence by Identity (2.6) we have $V^z \pi(f) V^{-z} \in C(Z)$ and moreover by construction, if $x \in X$ and $y = \theta(x)$, since θ is an orbit injection with cocycle η:

$$
\begin{aligned}
V^z \pi(f) V^{-z}(y) & = \left(\sum_{h \in \mathbb{Z}^d} p_h^z \pi(f) \circ \psi^{-h} \right)(y) \\
& = p_{\eta(x,z)}^z(y) \pi\left(f \circ \varphi^{-z} \right)(y) \\
& = \pi\left(f \circ \varphi^{-z} \right)(y).
\end{aligned}
$$

On the other hand, if $y \notin \theta(X)$ then:

$$
\begin{aligned}
V^z \pi(f) V^{-z}(y) & = \left(\sum_{h \in \mathbb{Z}^d} p_h^z \pi(f) \circ \psi^{-h} \right)(y) \\
& = 0 \\
& = \pi(f \circ \varphi^{-z})(y)
\end{aligned}
$$

where the last equality follows from $\pi(g)(Z/X') = \{0\}$ for all $g \in C(X)$ by construction.

Hence the pair (π, V) is covariant for $\left(X, \varphi, \mathbb{Z}^d \right)$. It is however degenerate, and its integrated *-morphism is actually valued in the corner $p \left(C(Z) \rtimes_\psi \mathbb{Z}^d \right) p$. It will be convenient to work explicitly in this corner. We note that for all $f \in C(X)$ we have $\pi(f) \in p C(Z) p = C\left(\theta(X)\right)$ and p commutes with V^z for all $z \in \mathbb{Z}^d$. Hence, setting $\pi' = p \pi\,(\cdot)\, p$ and $\omega^g = p V^g p$ for $g \in \mathbb{Z}^d$ we define a nondegenerate covariant pair (π', ω) valued in $p \left(C(Z) \rtimes_\psi \mathbb{Z}^d \right) p$. By [16, Proposition 2.39] there exists a unique *-morphism α such that for all $f \in C(X)$ we have $\alpha(f) = \pi'(f) \in p C(Z) p$ and for all $z \in \mathbb{Z}^d$ we have $\alpha(U_\varphi^z) = \omega^z$. Since $(X, \varphi, \mathbb{Z}^d)$ is minimal, the morphism α is injective. We now investigate the range of α. First by construction, the range of α is a C*-subalgebra of $p(C(Z) \rtimes_\psi \mathbb{Z}^d)p$ and $\pi'(1) = \alpha(1) = p$. Second, $\pi' = \alpha_{|C(X)}$ is a *-isomorphism onto $C\left(\theta(X)\right) \subseteq C(Z)$. Now, let $h \in \mathbb{Z}^d$. Set $X_h = \left\{ x \in X' : \psi^h(x) \in X' \right\}$ and note that X_h is clopen in X'. Let q_h be the

indicator function of X_h in $C(Z)$. Now:

$$\left((p - q_h) U_\psi^h (p - q_h)\right)^* \left((p - q_h) U_\psi^h (p - q_h)\right) = (p - q_h) \cdot (p - q_h) \circ \psi^h.$$

Note that $p - q_h$ is a projection (since q_h is a subprojection of p). Now, if $(p - q_h)(x) = 1$ then $x \in \theta(X)$ and $\psi^h(x) \notin \theta(X)$ and thus $(p - q_h)(\psi^h(x)) = 0$. Hence $(p - q_h) U_\psi^h (p - q_h) = 0$. Similarly:

$$\left(q_h U_\psi^h (p - q_h)\right)^* \left(q_h U_\psi^h (p - q_h)\right) = (p - q_h) \cdot q_h \circ \psi^h = 0$$

and thus $q_h U_\psi^h (p - q_h) = 0$. We would prove $(p - q_h) U_\psi^h q_h = 0$ the same way. We conclude:

$$p U_\psi^h p = q_h U_\psi^h q_h.$$

On the other hand, let $x \in X_h$ and let $w \in X$ be the unique element such that $x = \theta(w)$. Since θ is an orbit injection, we conclude that there exists $k \in \mathbb{Z}^d$ such that $\theta(\varphi^{-k}(w)) = \psi^{-h}(x)$, i.e. $p_h^k(x) = 1$. Since θ, ψ and φ are free we conclude that if $k' \neq k$ then $p_h^{k'}(x) = 0$: indeed, assume there exists $k' \in \mathbb{Z}^d$, $w', w'' \in X$ such that $x = \theta(w')$, $w' = \varphi^{k'}(w'')$ and $x = \psi^h(\theta(w''))$. Then since θ is injective, $w' = w$. Since ψ^h and θ are injective, $w'' = \varphi^{-k}(w)$. Hence $\varphi^{-k'}(w) = \varphi^{-k}(w)$ and since φ is free we conclude that $k = k'$. Hence the projections p_h^j for $j \in \mathbb{Z}^d$ are pairwise orthogonal and thus the sets X_h^k ($k \in \mathbb{Z}^d$) are disjoint.

Since X_h is compact and X_h is the disjoint union of the clopen subsets X_h^k ($k \in \mathbb{Z}^d$) of X_h, we also conclude that the set $\left\{k \in \mathbb{Z}^d : p_h^k \neq 0\right\}$ is finite. Therefore we can write:

$$q_h = \sum_{k \in \mathbb{Z}^d} p_h^k \in C(Z).$$

Now, for all $z, z' \in \mathbb{Z}^d$ we have by construction: $p_h^z \omega^z p_h^{z'} = p_h^z U_\psi^h p_h^{z'}$ so (noting all the sums are finite):

$$\begin{aligned} p U_\psi^h p &= q_h U_\psi^h q_h = \sum_{z \in \mathbb{Z}^d} \sum_{z' \in \mathbb{Z}^d} p_h^z U_\psi^j p_h^{z'} \\ &= \sum_{z \in \mathbb{Z}^d} \sum_{z' \in \mathbb{Z}^d} p_h^z \omega^z p_h^{z'} \in \operatorname{ran} \alpha. \end{aligned}$$

Hence the range of α is $p(C(Z) \rtimes_\psi \mathbb{Z}^d)p$ as claimed. $\qquad \square$

We now proceed to prove the converse of Proposition (2.2) and thus establish our main theorem for this section. We use the following notations. Let G be an Abelian discrete group and let φ be an action of G on a compact space X. We denote the Pontryagin dual of G by $\widehat{G}$. We define the dual action γ of the compact group $\widehat{G}$ on $C(X) \rtimes_\varphi G$ as the unique action such that for all $f \in C(X), \chi \in \widehat{G}$ and $g \in G$ we have $\gamma^\chi(f) = f$ and $\gamma^\chi(U^g) = \chi(g)U^g$. In this context, we define, for all $a \in C(X) \rtimes_\varphi G$:

$$\text{(2.7)} \qquad \mathbb{E}(a) = \int_G \gamma^\chi(a) d\mu(\chi)$$

where μ is the Haar probability measure on the compact group $\widehat{G}$. It is standard that $\mathbb{E}$ is a conditional expectation from $C(X) \rtimes_\varphi G$ onto the fixed point of γ which is $C(X)$. We refer to [17], [9] or [16] for the proof of the existence of the strongly continuous action γ and its fundamental properties.

We have the following immediate extension of [12, Lemma 5.1] from actions of $\mathbb{Z}$ to actions of $\mathbb{Z}^d$:

Lemma 2.3. *Let $d \in \mathbb{N} \setminus \{0\}$. Let X be an infinite compact space and let φ be a minimal free action of $\mathbb{Z}^d$ on X by homeomorphisms. Let v be a unitary in $C(X) \rtimes_\varphi \mathbb{Z}^d$ such that $vC(X)v^* = C(X)$. Then there exists orthogonal projections $(p_g)_{g \in \mathbb{Z}^d}$ in $C(X)$ such that $\sum_{g \in \mathbb{Z}^d} p_g = 1$ while $\{g \in \mathbb{Z}^d : p_g \neq 0\}$ is finite and:*

$$v = f \sum_{g \in \mathbb{Z}^d} p_g U_\varphi^g$$

for some $f \in C(X)$.

Proof. Let $G = \mathbb{Z}^d$ and let the Pontryagin dual of G be denoted by $\widehat{G}$ (i.e. $\widehat{G} = \mathbb{T}^d$). For $g \in G$ let δ_g be the Dirac measure at $g \in G$ (we identify measures with their density against the counting measure over the countable space G). Let $x \in X$. Define the following representation π_x of $C(X) \rtimes_\varphi G$ on $\ell^2(G)$: for $f \in C(X)$ we set $\pi_x(f)\delta_g = f(\varphi^{-g}(x))\delta_g$ and for $h \in G$ we set $\pi_x(U_\varphi^h)\delta_g = \delta_{g+h}$. This representation is known as the regular representation induced by the measure Dirac measure at x on X [9]. It is routine to check that $\pi_x(U_\varphi^h)\pi_x(f)\pi_x(U_\varphi^{-h}) = \pi_x(f \circ \varphi^{-h})$ and thus π_x extends uniquely to a *-representation of $C(X) \rtimes_\varphi G$. Moreover, since the action of G is minimal and X is infinite, the crossed-product $C(X) \rtimes_\varphi G$ is simple and thus π_x is faithful. In addition, π_x is irreducible (using the freedom of the action φ). These facts can be found in [9] and are well-known.

Let $g \in G$. Set $p_g = |\mathbb{E}(vU^{-g})|$ where $\mathbb{E}$ is the conditional expectation on $C(X) \rtimes_\varphi G$ defined by the dual action γ of $\widehat{G}$ and Identity (2.7). Let $X_g \subseteq X$ be the support of $p_g \in C(X)$. Let $\chi \in \widehat{G}$. We define the unitary u_χ on $\ell^2(G)$ by $u_\chi \delta_h = \chi(h)\delta_h$ for all $h \in G$. For $f \in C(X)$ we have:

$$\begin{aligned}
\pi_x(\gamma^\chi(f)) &= \pi_x(f) \\
&= u_\chi \pi_x(f) u_\chi^*
\end{aligned}$$

(since u_χ is diagonal in the basis $(\delta_g)_{g \in G}$ so commutes with the diagonal operators $\pi_x(f)$ for all $f \in C(X)$). Let $g \in G$. Then:

$$\begin{aligned}
u_\chi \pi_x(U^g) u_\chi^* \delta_h &= u_\chi \pi_x(U^g)\overline{\chi(h)}\delta_h \\
&= u_\chi \pi_x(U^g)\chi(-h)\delta_h \\
&= u_\chi \chi(-h+g)\delta_{h+g} \\
&= \chi(g)\delta_{h+g} \\
&= \chi(g)\pi_x(U^g)\delta_h \\
&= \pi_x(\gamma^\chi(U^g))\delta_h.
\end{aligned}$$

Hence since π_x is a continuous *-morphism, $\pi_x \circ \gamma^\chi = \operatorname{Ad} u_\chi \circ \pi_x$. (Note that since the action is minimal, π_x is actually injective on $C(X)$ and together with the intertwining of the dual actions, this fact ensures an alternative proof that π_x is faithful).

Since $\pi_x(v)$ stabilizes $\pi_x(C(X))$, it also stabilizes $\pi_x(C(X))''$. Now:

$$\pi_x(C(X))'' = \ell^\infty(G)$$

where $\ell^\infty(G)$ is identified with the multiplication operators on $\ell^2(G)$ or, equivalently, with the maximal Abelian Von Neumann algebra of diagonal operators in

the basis $\{\delta_g : g \in G\}$. Indeed, the inclusion $\ell^\infty(G) \subseteq \pi_x\left(C(X)\right)'$ is easily checked, and if $T \in \mathcal{B}\left(\ell^2\left(G\right)\right)$ with $\langle T\delta_g, \delta_{g'}\rangle \neq 0$ for some $g \neq g'$ then, choosing $f \in C(X)$ so that $f(\alpha_g(x)) \neq f(\alpha_{g'}(x))$ we conclude that:

$$
\begin{aligned}
\langle T\pi(f)\delta_g, \delta_{g'}\rangle &= f(\alpha_g(x))\langle T\delta_g, \delta_{g'}\rangle, \\
\langle \pi(f)T\delta_g, \delta_{g'}\rangle &= f\left(\alpha_{g'}(x)\right)\langle T\delta_g, \delta_{g'}\rangle,
\end{aligned}
$$

and thus $\pi(f)T \neq T\pi(f)$. Hence, $T \in \pi(C(X))'$ if and only if $\varphi \in \ell^\infty(G)$. Hence $\ell^\infty(G) = \pi_x\left(C(X)\right)'$ and thus $\pi_x(C(X))'' = \ell^\infty(G)' = \ell^\infty(G)$.

Thus there exists $(\lambda_h)_{h \in G}$ with $\lambda_h \in \mathbb{T}$ and $\sigma : G \to G$ a permutation such that for all $h \in G$:

$$
\pi_x(v)\delta_h = \lambda_h \delta_{\sigma(h)}.
$$

(note: if Q_h is the projection onto $\mathbb{C}\delta_h$ then $\pi_x(v)Q_h\pi_x(v)^* \in \ell^\infty(G)$ and is a projection so $\pi_x(v)Q_h\pi_x(v)^* = Q_{\sigma(h)}$.)

Then for all $g \in G$ and $h \in G$ we have:

$$
\begin{aligned}
\pi_x\left(\mathbb{E}_g(v)\right)\delta_h &= \int_{\widehat{G}} \pi_x\left(\gamma^\chi(\nu U^{-g})\right)\delta_h d\mu(\chi) \\
&= \int_{\widehat{G}} u_\chi \pi_x\left(v\right)\pi_x\left(U^{-g}\right) u_\chi^* \delta_h d\mu(\chi) \\
&= \int_{\widehat{G}} \chi\left(\sigma\left(h - g\right) - h\right)\lambda_{h-g}\delta_{\sigma(h-g)} d\mu\left(\chi\right) \\
&= \begin{cases} \lambda_{g-h}\delta_h & \text{if } \sigma(h-g) = h \\ 0 & \text{if } \sigma(h-g) \neq h \end{cases}.
\end{aligned}
$$

Hence:

$$
\pi_x\left(p_g\right)\delta_h = \begin{cases} \delta_h & \text{if } \sigma(h - g) = h, \\ 0 & \text{if } \sigma(h - g) \neq h. \end{cases}
$$

This proves that p_g is a projection and $p_g p_h = 0$ iff $g \neq h \in G$. In particular, X_g is clopen since p_g continuous.

Let $-g_x = \sigma^{-1}(0)$ (note: σ depends on x as it is defined via $\pi_x(v)$). Then if $0 \in G$ is the neutral element of G:

$$
p_{g_x}(x) = \langle \pi_x\left(p_{g_x}\right)\delta_0, \delta_0\rangle = 1.
$$

Since x is arbitrary in X, we conclude that $\bigcup_{g \in G} X_g = X$ (by above equation: if $x \in X$ then $x \in X_{g_x}$). Since $\{X_g : g \in G\}$ is an open covering of the compact X there exists a finite subset $S \subseteq G$ such that $X = \bigcup_{g \in S} X_g$. Since the sets X_g are pairwise disjoint (as $p_g p_h = 0$) we conclude that if $g \in S^c$ then $X_g = \emptyset$. Hence, $p_g = 0$ if $g \in S^c$ and $\oplus_{g \in S} p_g = 1$.

Last, let $v_0 = \sum_{g \in S} p_g U^g$. By construction, $\pi_x\left(vv_0^*\right) = \lambda$. Since π_x is faithful and unital and λ is a unitary, so is vv_0^* and thus v_0 is a unitary. In particular, $\sum_{g \in G} \varphi^{-g}(p_g) = v_0^* v_0 = 1$. Moreover, since $\lambda \in \pi_x\left(C(X)\right)''$ and $C(X)$ is maximal Abelian in $C(X) \rtimes_\varphi G$ (as φ is free), we conclude that there exists $f \in C(X)$ such that $\pi_x(f) = \lambda$. By faithfulness of π_x we conclude that $v = fv_0$ as claimed. $\qquad \square$

We now prove the main result of this section:

Theorem 2.4. *Let $\left(X, \varphi, \mathbb{Z}^d\right)$ and $\left(Z, \psi, \mathbb{Z}^d\right)$ be two free minimal Cantor dynamical systems. The following are equivalent:*

(1) *There exists a bounded orbit injection $\theta : X \to Z$,*
(2) *There exists a *-monomorphism $C(X) \rtimes_\varphi \mathbb{Z}^d$ into $C(Z) \rtimes_\psi \mathbb{Z}^d$ such that:*
   - *$\alpha(C(X)) \subseteq C(Z)$,*
   - *Letting $p = \alpha(1)$, the range of α is $p\left(C(Z) \rtimes_\psi \mathbb{Z}^d\right)p$.*

Proof. Proposition (2.2) establishes that (1) implies (2). We are left to show that (2) implies (1).

Assume henceforth that we are given a *-monomorphim $\alpha : C(X) \rtimes_\varphi \mathbb{Z}^d \to C(Z) \rtimes_\psi \mathbb{Z}^d$ with the properties mentioned in (2). Let $p = \alpha(1) \in C(Z)$ (p is a continuous $\{0, 1\}$ valued function on Z whose support is a clopen subset of Z denoted by X'). Note that we can write any operator in $C(Z) \rtimes_\psi \mathbb{Z}^d$ as a 2 by 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a \in p\left(C(Z) \rtimes_\psi \mathbb{Z}^d\right)p$, $b \in p\left(C(Z) \rtimes_\psi \mathbb{Z}^d\right)(1 - p)$, $c \in (1 - p)\left(C(Z) \rtimes_\psi \mathbb{Z}^d\right)p$ and $d \in (1 - p)\left(C(Z) \rtimes_\psi \mathbb{Z}^d\right)(1 - p)$. Now, in this decomposition, for all $g \in \mathbb{Z}^d$ we set $\omega^g = \alpha(U_\varphi^g)$ and $V^g = \begin{bmatrix} \omega^g & 0 \\ 0 & 1 - p \end{bmatrix}$. Fix $g \in \mathbb{Z}^d$. By assumption on α we have:

$$C(Z) = \left\{ \begin{bmatrix} \alpha(f) & 0 \\ 0 & f' \end{bmatrix} : f \in C(X), f' \in C(Z \backslash X') \right\}.$$

Fix $g \in \mathbb{Z}^d$. We also have by our assumptions on α that for all $f \in C(X)$:

$$\omega^g \alpha(f) \omega^{-g} = \alpha(U_\varphi^g f U_\varphi^{-g}) = \alpha(f \circ \varphi_{-g}) \in pC(Z)p.$$

Hence V^g stabilizes $C(Z)$ in $C(Z) \rtimes_\psi \mathbb{Z}^d$. By Lemma 2.3, we conclude that for all $h \in \mathbb{Z}^d$ there exists a projection $p_h^g \in C(Z)$ such that $\sum_{h \in \mathbb{Z}^d} p_h^g = 1$ with $\left\{ h \in \mathbb{Z}^d : p_h^g \neq 0 \right\}$ finite and $f_g \in C(X')$ (valued in $\mathbb{T}$) such that:

$$V^g = f_g \sum_{h \in \mathbb{Z}^d} p_h^g U_\psi^h.$$

By assumption on α, the set X' is homeomorphic to X. Then let $x \notin X'$. Let $h \in \mathbb{Z}^d \backslash \{0\}$. Then if $p_h^g(x) \neq 0$ (hence $= 1$) then $p_k^g(x) = 0$ for all $k \neq h$ by orthogonality. Let $f' \in C(Z)$ supported in $Z \backslash X'$ and such that $1 = f'(x) \neq f'\left(\psi^{-h}(x)\right)$. Then $V^g f' V^{-g} = f'$ by construction. Yet:

$$\left(f_g \sum_{k \in \mathbb{Z}^d} p_k^g U_\psi^k \right) f' \left(f_g \sum_{k \in \mathbb{Z}^d} p_k^g U_\psi^k \right)^*$$

$$= f_g \left(\sum_{k \in \mathbb{Z}^d} \sum_{k' \in \mathbb{Z}^d} \left(p_k^g \left(f' \circ \psi^{-k} \right) \left(U_\psi^{k - k'} \right) p_{k'}^g \right) \right) f_g^*$$

$$= f_g \left(\sum_{k \in \mathbb{Z}^d} \sum_{k' \in \mathbb{Z}^d} \left(f' \circ \psi^{-k} \right) \left(p_k^g p_{k'}^g \circ \psi^{k' - k} \right) U_\psi^{k - k'} \right) f_g^*,$$

and thus, by orthogonality, as in Identity (2.6) in the proof of Proposition (2.2), we see that:

$$\left(f_g \sum_{k \in \mathbb{Z}^d} p_k^g U_\psi^k \right) f' \left(f_g \sum_{k \in \mathbb{Z}^d} p_k^g U_\psi^k \right)^* = \sum_{k \in \mathbb{Z}^d} p_k^g \cdot f' \circ \psi^{-k}.$$

Hence:

$$\left(\sum_{k \in \mathbb{Z}^d} p_k^g U_\psi^k \right) f' \left(\sum_{k \in \mathbb{Z}^d} p_k^g U_\psi^k \right)^* (x) = p_h^g(x) \cdot f' \circ \psi^{-h}(x) \cdot p_h^g(x)$$

$$= f' \circ \psi^{-h}(x) \neq f'(x).$$

This is a contradiction. Hence $p_h^g(x) = 0$ and $p_h^g \in C(X')$ for all $h \neq 0$ and $g \in G$. Clearly $1 - p$ is a subprojection of p_0^g for all $g \in G$. It will be convenient to change our notation in the sequel, and denote the projection $p_0^g p$ by p_0^g for all $g \in \mathbb{Z}^d$. With this new notation, $\sum_{h \in \mathbb{Z}^d} p_h^g = p$.

Let $\theta : X \to X'$ be the homeomorphism defined by $\alpha(f)(x) = f \circ \theta^{-1}(x)$ for all $x \in X'$ and $f \in C(X)$. We claim that θ, identified as an injection $X \to Z$, is a bounded orbit injection.

Let $x \in X$ and $g \in \mathbb{Z}^d$. Let X_h^g be the support of p_h^g for all $h \in \mathbb{Z}^d$ and note that X_h^g is a clopen subset of X'. Since $\sum_{h \in \mathbb{Z}^d} p_h^g = p$ and these projections are orthogonal, there exists a unique $h = h(x, g) \in \mathbb{Z}^d$ such that $\theta(\varphi^g(x)) \in X_h^g$. Now, since there are only finitely many nonzero projections $\{p_h^g : h \in \mathbb{Z}^d\}$, the map $x \mapsto h(x, g)$ is bounded on X.

Let $x, y \in X$ and set $z = \theta(y)$. Then $\varphi^g(x) = y$ implies $z \in X_{h(x,g)}^g$. Let $h = h(x, g)$. Now by construction, $z \in X_{h(x,g)}^g$ if and only if $p_k^g(z) = 0$ for all $k \in \mathbb{Z}^d \setminus \{h\}$. Using the same computation technique as before, we have:

$$\alpha(f) \circ \psi^{-h(x,g)}(z) = U_\psi^{h(x,g)} \alpha(f) U_\psi^{-h(x,g)}(z)$$

$$= \left(\sum_{k \in \mathbb{Z}^d} p_k^g U_\psi^k \right) \cdot \alpha(f) \cdot \left(\sum_{k \in \mathbb{Z}^d} p_k^g U_\psi^k \right)^* (z)$$

$$= \left(V^g \alpha(f) V^{-g} \right)(z)$$

$$= \alpha(U_\varphi^g f U_\varphi^{-g})(z)$$

(2.8) $$= \alpha \left(f \circ \varphi_{-g} \right)(z).$$

Hence $\varphi^g(x) = y$ implies by definition of α and Equality (2.8):

$$f \circ \varphi^{-g}(y) = f \circ \theta^{-1} \circ \psi^{-h(x,g)} \circ \theta(y)$$

for all $f \in C(X)$, or equivalently as $C(X)$ separates the points of X:

$$x = \varphi^{-g}(y) = \theta^{-1} \circ \psi^{h(x,g)} \circ \theta(y),$$

or equivalently: $\psi^{-h(x,g)}(\theta(x)) = \theta(y)$.

Let us now assume that $\psi^k(x) = y$ for some $x, y \in X'$ and $k \in \mathbb{Z}^d$. We shall use the following sequence of claims to establish the existence of $n(x, g) \in \mathbb{Z}^d$ such that $\varphi^{n(x,g)}(\theta^{-1}(x)) = \theta^{-1}(y)$.

Let $A = C(Z) \rtimes_\psi \mathbb{Z}^d$. The dual action γ of $\mathbb{T}^d$ on A defines spectral subspaces by setting:

$$\forall g \in \mathbb{Z}^d \quad A_g = \left\{ a \in A : \forall \omega \in \mathbb{T}^d \quad \gamma^\omega(a) = \omega^g a \right\}$$

where, if we write $\omega = (\omega_1, \ldots, \omega_d)$ and $g = (g_1, \ldots, g_d)$ then $\omega^g = (\omega_1^{g_1}, \ldots, \omega_d^{g_d})$. For $a, b \in A$ we set:

$$\langle a, b \rangle = \mathbb{E}\left(b^* a \right)$$

and we thus defined an $C(Z)$-valued inner product. We check easily that A_k and A_m are orthogonal for $k \neq m$ for this inner product.

We now prove:

Claim 2.5. *There exists $x \in X'$ and $k \in \mathbb{Z}^d$ with $\psi^k(x) \in X'$ if and only if $pU_\psi^k p \neq 0$.*

Note that:

$$\left(pU_\psi^k p \right)^* \left(pU_\psi^k p \right) = p \left(p \circ \psi^k \right) p$$

$$(2.9) \qquad\qquad\qquad\qquad = p \left(p \circ \psi^k \right).$$

Hence $pU_\psi^k p \neq 0$ if and only if $\left(pU_\psi^k p \right) \left(pU_\psi^k p \right)^* \neq 0$ which is equivalent by (2.9) to the existence of $x \in X'$ such that $\psi^k(x) \in X'$.

Claim 2.6. *For any $k \in \mathbb{Z}^d$ we have $pU_\psi^k p \in A_k$.*

Let $\chi \in \mathbb{T}^d$. Then, note that $p = \alpha(1) \in C(Z)$ by assumption so $\gamma^\chi(p) = p$ for all $\chi \in \mathbb{T}^d$ hence:

$$\gamma^\chi(pU_\psi^k p) = \gamma^\chi(p)\gamma^\chi(U_\psi^k)\gamma^\chi(p) = p\gamma^\chi(U_\psi^k)p = \chi^k pU_\psi^k p$$

as claimed.

Claim 2.7. *Suppose that there exists $k \in \mathbb{Z}^d$ such that for all $g \in \mathbb{Z}^d$ and $x \in X'$ we have $p_k^g(x) = 0$. Then $\alpha(C(X) \rtimes_\varphi \mathbb{Z}^d)$ is orthogonal to A_k for $\langle ., . \rangle$.*

The C*-algebra $\alpha(C(X) \rtimes_\varphi \mathbb{Z}^d)$ is generated by $\alpha(C(X)) \subseteq C(Z)$ and ω^g ($g \in \mathbb{Z}^d$). By Lemma 2.3, for all $g \in \mathbb{Z}^d$ we have $\omega^g = f_g \sum_{h \in \mathbb{Z}^d} p_h^g U_\psi^h$ after replacing p_0^g with $p_0^g p$ and with $f_g \in C(X')$. Thus with our assumption, $\omega^g \perp A_k$. Hence $\alpha(C(X) \rtimes_\varphi \mathbb{Z}^d)$ is orthogonal to A_k.

Claim 2.8. *Suppose that there exists $x \in X'$ and $k \in \mathbb{Z}^d$ such that $y := \psi^k(x) \in X'$. Assume moreover that there is no $g \in \mathbb{Z}^d$ such that $\theta^{-1}(y) = \varphi^g(\theta^{-1}(x))$. Then for all $g \in \mathbb{Z}^d$ and $z \in X'$ we have $p_k^g(z) = 0$.*

Let $g, j \in \mathbb{Z}^d$. By definition, if $p_k^g \left(\theta \circ \varphi^j \circ \theta^{-1}(x) \right) \neq 0$ then

$$y = \theta(\varphi^g(\theta^{-1}(\theta \circ \varphi^j \circ \theta^{-1}(x)))) = \theta \left(\varphi^{g+j} \left(\theta^{-1}(x) \right) \right).$$

So by assumption $p_k^g \left(\theta \circ \varphi^j \circ \theta^{-1}(x) \right) = 0$ for all $g, j \in \mathbb{Z}^d$. Now:

$$\overline{\left\{ \varphi^j \left(\theta^{-1}(x) \right) : j \in \mathbb{Z}^d \right\}} = X$$

since φ is minimal.

Since θ and p_k^g are continuous and $p_k^g \circ \theta$ is null on $\left\{ \varphi^j \left(\theta^{-1}(x) \right) : j \in \mathbb{Z}^d \right\}$, and since $\theta(X) = X'$, we conclude that $p_k^g(X') = \{0\}$ for all $g \in \mathbb{Z}^d$. Since p_k^g is supported on X' by construction, we conclude that p_k^g is null on X' for all $g \in \mathbb{Z}^d$ as claimed.

Claim 2.9. *If there exists $x \in X'$ such that $\psi^k(x) = y \in X'$ then there exists $g \in \mathbb{Z}^d$ such that $\varphi^g(\theta^{-1}(x)) = \theta^{-1}(y)$.*

By Claim (2.5), we conclude $pU_\psi^k p \neq 0$. Since α is onto pAp, we conclude $pU_\psi^k p \in \operatorname{ran} \alpha$. Yet $pU_\psi^k p \in A_k$ by Claim (2.6). If we assume that there exists no $g \in \mathbb{Z}^d$ such that $\varphi^g(x) = y$ then by Claim (2.8) $p_k^g = 0$ for all $g \in \mathbb{Z}^d$. By Claim (2.7), this would imply that $\operatorname{ran} \alpha$ is orthogonal to A_k. We have reached a contradiction. Hence there exists $g \in \mathbb{Z}^d$ such that $\theta \circ \varphi^g \circ \theta^{-1}(x) = y$.

Hence we conclude that θ is a bounded orbit injection as desired.   □

We note that our result proves, among other things, that the corner $p\left(C(Z) \rtimes_\psi \mathbb{Z}^d\right) p$ with the notation of Theorem (2.4) is in fact a C*-crossed-product itself, a fact which follows in our proof from the special nature of bounded orbit injections.

The ordered K_0 group of C*-algebras is an invariant for Rieffel-Morita equivalence, and Theorem (2.4) implies that two minimal free Cantor systems are bounded orbit injection equivalence must have Rieffel-Morita equivalent C*-crossed-products. Hence the ordered K_0 group of C*-crossed-products of minimal free Cantor systems is an invariant of bounded orbit injection equivalence. However the computation of the K_0 group of a C*-crossed-product of a minimal free actions of $\mathbb{Z}^d$ on a Cantor set is a delicate matter for $d > 1$, as shown for instance in [10] – unless in the case $d = 1$ where the Pimsner-Voiculescu six-term exact sequence suffices [1]. As we shall see in the next section, one can however consider an alternative ordered group as an invariant for orbit injection equivalence, hence for the equivalence described in Theorem (2.4) between C*-crossed-products of minimal free Cantor actions.

3. Orbit Equivalence from Orbit Injection Equivalence

If $\left(X, \varphi, \mathbb{Z}^d\right)$ and $\left(Y, \psi, \mathbb{Z}^d\right)$ are bounded orbit injection equivalent, there is a third system $\left(Z, \alpha, \mathbb{Z}^d\right)$ and bounded orbit injections from both $\left(X, \varphi, \mathbb{Z}^d\right)$ and $\left(Y, \psi, \mathbb{Z}^d\right)$ into $\left(Z, \alpha, \mathbb{Z}^d\right)$. In this section, we focus our attention on the problem of omitting the system $\left(Z, \alpha, \mathbb{Z}^d\right)$. That is, we address the question of when the existence of a bounded orbit injection equivalence be strengthened to the existence of a bounded orbit injection and/or an orbit equivalence.

A particularly useful invariant of bounded orbit equivalence for our purpose is an ordered group defined by:

$$\begin{aligned} G\left(\varphi\right) &= C\left(X, \mathbb{Z}\right) / \langle f - f\varphi^v : v \in \mathbb{Z}^d \rangle \\ G\left(\varphi\right)_+ &= \{[f] : f\left(x\right) \geq 0 \text{ for all } x \in X\}. \end{aligned}$$

This ordered group has been studied for minimal $\mathbb{Z}^d$ systems, e.g. in [4, 8], and is referred to there as the dynamical cohomology group. For $d = 1$, the triple formed by this ordered group along with a distinguished order unit forms a complete invariant for strong orbit equivalence. What, if any, analogous theorem there is for $d > 1$ is an open problem.

In [7], Theorem (2.1) proves that if the suspension spaces of two minimal free Cantor systems $\left(X, \varphi, \mathbb{Z}^d\right)$ and $\left(Y, \psi, \mathbb{Z}^d\right)$ are homeomorphic then $\left(X, \varphi, \mathbb{Z}^d\right)$ and $\left(Y, \psi, \mathbb{Z}^d\right)$ are bounded-orbit injection equivalent. The converse is proven for $d \in \{1, 2\}$ in [7]. Thus, one can apply some results proven in this section if we know

that two minimal free Cantor systems $\left(X, \varphi, \mathbb{Z}^d\right)$ and $\left(Y, \psi, \mathbb{Z}^d\right)$ are suspension equivalent. We work here with the simpler relation of bounded orbit injection equivalence.

3.1. **Voronoi-Rohlin Partitions.** We create Voronoi-Rohlin partitions here, using techniques similar to those used in [4, 7, 13, 14], and analogous to the way that Rohlin tower partitions are used in [12, Theorem 2.6] for $\mathbb{Z}$-actions. More sophisticated results are in [14], we include the necessary arguments here for completeness. We use Voronoi tilings to tile $\mathbb{Z}^d$ for each point x, thus partitioning the points in each φ-orbit into equivalence classes. The idea is that for two points x, y in the same equivalence class, we can determine a vector $v(x) \in \mathbb{Z}^d$ such that $\varphi^{v(x)}(x) = y$. Since we define $v(x)$ in a locally constant way, we are able to define a continuous cocycle $v(x)$ to serve our purposes.

We begin with some basics of Voronoi tilings associated with an M-regular set $R \subset \mathbb{R}^d$. Below, we denote the Euclidean metric in $\mathbb{R}^d$ by d , and the Euclidean norm by $\|.\|$.

Definition 3.1. *Let $R \subset \mathbb{Z}^d$ and $M > 0$. We say that R is M-regular if*

    (1) *R is M-separated, that is, for any $v \in R$, if $w \in R$ and $w \neq v$ then $d(v, w) > M$.*

    (2) *R is $2M$-syndetic, that is, for any $v \in \mathbb{R}^d$, there is a $w \in R$ such that $d(v, w) < 2M$*

Specifically, suppose $R \subset \mathbb{Z}^d$ is an M-regular set, and $M > 2$. For each $p \in \mathbb{R}^d$, let $v(p)$ be the (typically singleton) set of points $w \in R$ which achieve $\min_{w \in R} d(p, w) = d(p, v(p))$. Note that $\min_{w \in R} d(p, w)$ is well-defined and uniformly bounded by $2M$ because the set R is $2M$-syndetic. For $w \in R$, the tile containing w is the set $T(w) = \left\{p \in \mathbb{R}^d : w \in v(p)\right\}$. The covering of $\mathbb{R}^d$ by the tiles $\{T(w) : w \in R\}$ is what we refer to as the *Voronoi tiling $\tau(R)$*.

We note that with this setup, each tile $T(w)$ is a convex, compact subset of $\mathbb{R}^d$ which is the intersection of a finite number of closed half-spaces. Because $R \subset \mathbb{Z}^d$ and R is $2M$-syndetic, there are only finitely many tiles up to translation. That is, there are only finitely many different sets $P_1, P_2, \ldots, P_k$ of the form $T(w) - w$ where $w \in R$, The sets P_i are referred to as the *prototiles* of the Voronoi tiling $\tau(R)$.

The following will serve as an important preliminary result because it will later be used to bound the number of vectors that are near the boundary of a tile.

Lemma 3.2. *Let $d \geq 1$. Then there is a $b > 0$ (depending only on d) such that for any $M > 2$ and any M-regular set $R \subset \mathbb{Z}^d$ if $\tau(R)$ is the Voronoi tiling of $\mathbb{R}^d$ associated with R then any tile in τ intersects at most b other tiles.*

Proof. Let $B(x, r) \subset \mathbb{R}^d$ denote the ball in $\mathbb{R}^d$ centered at x and with radius r. Let b be the maximum cardinality of a set $\{y_1, y_2, \ldots, y_n\}$ such that $y_i \in B(0, 2)$ for all i, and $B(y_i, 1/2) \cap B(y_j, 1/2) = \emptyset$ for $i \neq j$.

Now suppose $M > 2$ and that $R \subset \mathbb{Z}^d$ is an M-regular set. Let T_0 be a tile in $\tau(R)$, and let $a(T_0) = \{T_1, T_2, \ldots, T_n\}$ be the set of tiles in $\tau(R)$ which intersect, but are not equal to T_0. Let $x_i \in R$ denote the center of T_i for $0 \leq i \leq n$. A tile T_i intersects T_j if there is a point $p \in \mathbb{R}^d$ such that $d(p, x_i) = d(p, x_j) = \min_{v \in R} d(p, v)$.

Because R is $2M$-syndetic, $d(x_0, x_i) < 4M$ for all $1 \leq i \leq n$. Because R is M-separated, $d(x_i, x_j) > M$ meaning that $B(x_i, M/2) \cap B(x_j, M/2) = \emptyset$ for any $1 \leq i, j \leq n$.

Now let $y_i = \frac{1}{M}(x_i - x_0)$. From the above it follows that the points y_i are in $B(0, 2)$ for $0 \leq i \leq n$ and $B(y_i, 1/2) \cap B(y_j, 1/2) = \emptyset$ for $0 \leq i < j \leq n$. Therefore $n \leq b$. $\qquad\square$

To construct partitions associated with Voronoi tilings, suppose $(X, \varphi, \mathbb{Z}^d)$ is a minimal Cantor system. Let $C \subset X$ be clopen, and for each $x \in X$ consider the set of return times of x to C: $R_C(x) = \{v \in \mathbb{Z}^d : \varphi^v(x) \in C\} \subset \mathbb{Z}^d$.

Definition 3.3. *Let $C \subset X$ be clopen. We say that C is M-regular if for all $x \in X$, the set $R_C(x)$ is M-regular.*

The following propositions establish the existence of M-regular clopen sets C with various properties.

Proposition 3.4. *Let $(X, \varphi, \mathbb{Z}^d)$ be a minimal Cantor system, and let $M > 2$. There is an $\varepsilon > 0$ such that if $C \subset X$ is clopen with $diam(C) < \varepsilon$ then C is M-separated.*

Proof. Suppose not. Then there is an $M > 2$ such that for every $n \geq 1$, there is a clopen set $C_n \subset X$, a point $x_n \in C_n$ and a vector $v_n \in \mathbb{Z}^d$ such that $diam(C_n) < \frac{1}{n}$, $\varphi^{v_n}(x_n) \in C_n$ and $0 < \|v_n\| < M$. There is a subsequence $\{x_{n_k}\}$ such that all vectors v_{n_k} are equal to a vector v with $0 < \|v\| < M$, and the sequence $\{x_{n_k}\}$ converges to a point $x_0 \in X$. Since φ^v is continuous, $\lim_{k \to \infty} \varphi^v(x_{n_k}) = \varphi^v(x)$. Since $d(\varphi^v(x_{n_k}), x_{n_k}) < \frac{1}{n_k}$, $\lim_{k \to \infty} \varphi^v(x_{n_k}) = x$, implying that $\varphi^v(x) = x$, which is a contradiction to the freeness of the action. $\qquad\square$

Proposition 3.5. *Let $(X, \varphi, \mathbb{Z}^d)$ be a minimal Cantor system. Let C be any clopen set. There is an $r > 0$ such that C is r-syndetic.*

Proof. Since φ is minimal, $\cup_{v \in \mathbb{Z}^d} \varphi^v C$ is an open cover of X. Because X is compact, there is a finite subcover. The result follows. $\qquad\square$

Proposition 3.6. *Let $(X, \varphi, \mathbb{Z}^d)$ be a minimal Cantor system. Let $C \subset X$ be any nonempty clopen set and $x_0 \in C$. Then there is an M_0 such that if $M > M_0$ then there exists a clopen set $D \subset C$ such that $x_0 \in D$ and D is M-regular.*

Proof. By the Proposition 3.5, there is an $r > 0$ such that C is r-syndetic. Let $M_0 = r$.

Assume that $M > M_0$. By Proposition 3.4, we may partition C into finitely many clopen sets of the form C_i, $1 \leq i \leq I$ where each C_i is M-separated. Without loss of generality, assume $x_0 \in C_1$.

Set $D_1 = C_1$ and for $1 < n \leq I$, recursively define:

$$D_n = D_{n-1} \cup \left(C_n \setminus \cup_{\|v\| \leq M} \varphi^v D_{n-1}\right).$$

Set $D = D_I$. Now suppose that $x, y \in D$ where $y = \varphi^w(x)$ with $w \neq 0$.

Case 1: Suppose $x, y \in C_i$ for some i. Then $\|w\| > M$ since each C_i is M-separated.

Case 2: Suppose $x \in C_i$, $y \in C_j$ for $j > i$. Then since $x \in D$, we have that $x \in D_i$ and therefore $x \in D_{j-1}$. Since $y \in D$ and $y \in C_j$ we have $y \notin \cup_{\|v\| \leq M} \varphi^v D_{j-1}$. Therefore, $\|w\| > M$ and D is M-separated.

Fix any $x \in X$. To show that D is $2M$-syndetic, it suffices to show that there is a vector w with $\|w\| < 2M$ such that $\varphi^w(x) \in D$.

Because C is M_0-syndetic, we know there is a vector w with $\|w\| < M_0 < M$ such that $\varphi^w(x) \in C$. Therefore, $\varphi^w(x) \in C_i$ for some i. If $\varphi^w(x) \in D_i \subset D$, we are done. Otherwise, $\varphi^w(x) \notin D_i$ which implies $\varphi^w(x) \in \cup_{\|v\| \leq M} \varphi^v D_{i-1}$. But then there is a v with $\|v\| \leq M$ such that $\varphi^w(x) \in \varphi^v D_{i-1}$ which implies $\varphi^{w-v}(x) \in D_{i-1} \subset D$ where $d(w, v) < M + M_0 < 2M$. $\qquad\square$

Now given $M > 2$, we can create a tiling for each point $x \in X$ in the following way. Fix an M-regular clopen set C and for $x \in X$ let $\tau(x, C) = \tau(R_C(x))$. There will only be finitely many different prototile sets $P_1, P_2, \ldots, P_K$ of the form $T(w) - w$ where $w \in R_C(x)$ even as we vary x over the entire space X. For $a \in C$, let $P(a)$ be the prototile containing the origin in $\tau(x, C)$. Fix a prototile P_k and set $C_k = \{a \in C : P(a) = P_k\}$. Then $C_k = \{a \in C : P(a) = P_k\}$ is clopen because $P(a)$ only depends upon the set $B(0, 4M) \cap R_C(a)$.

The above gives us a procedure for producing a certain clopen cover $\mathcal{A} = \{\varphi^w C_k : w \in P_k, 1 \leq k \leq K\}$ of X where $\cup_{k=1}^{K} C_k = C$. We note that given the above along with any finite clopen partition $\mathcal{P}$ of C, we will frequently take a partition

$$\mathcal{Q} = \{C_k : 1 \leq k \leq K\} \vee \mathcal{P}$$

of C, and consider the cover:

$$\mathcal{B} = \left\{ \varphi^w D_k^j : D_k^j \subset C_k, D_k^j \in \mathcal{Q}, w \in P_k, 1 \leq k \leq K \right\}$$

We will refer to this procedure as "refining $\{C_k\}$ if necessary". We note that after such a refinement, one may have $P(x) = P(y)$ for $x, y \in C$ in different partition elements. Points $x, y \in C$ in the same partition element will always have $P(x) = P(y)$.

What we really need from each tiling $\tau(x, C)$ is a partition of $\mathbb{Z}^d$. To this end, note that by refining $\{C_k\}$ if necessary, we may assume that if $x, \varphi^v(x) \in C_k$ for some k, then $\|v\| > 4M$, which insures that if x and $\varphi^v(x)$ are in the same partition element, the tile centered at the origin in $\tau(x, C)$ does not meet the tile centered at v. Now fix $x \in C$ and a tiling $\tau(x, X)$. For each $w \in R_C(x)$, consider the set $Z(w) = T(w) \cap \mathbb{Z}^d$, the tile centered at w intersected with $\mathbb{Z}^d$. The sets $\{Z(w) : w \in R_C(x)\}$ need not be pairwise disjoint. Thus for each $w \in R_C(x)$, we let $Z'(w)$ be the set of all elements of $Z(w)$ which do not have the property that $v \in Z(u)$ with $\varphi^w(x) \in C_k$, $\varphi^u(x) \in C_l$ and $l < k$. As a result for each $x \in X$, we obtain a partition $\mathcal{Z}(x) = \{Z'(w) : w \in R_C(x)\}$ of $\mathbb{Z}^d$. Again there are only finitely many different $\mathbb{Z}^d$-prototile sets $Z_1, Z_2, \ldots, Z_K$ of the form $Z'(w) - w$ where $w \in R_C(x)$ for $x \in X$, and for fixed k, the set of points x such that $Z'(0) \in \mathcal{Z}(x)$ is equal to Z_k is clopen.

Mostly inherent in the above discussion is the proof of the following theorem. Below, $A \triangle B$ denotes the symmetric difference of two sets A and B.

Theorem 3.7. *Let $(X, \varphi, \mathbb{Z}^d)$ be a minimal Cantor system. Let $x_0 \in C \subset X$ where C is clopen. Then there are integers $L \geq 1$ and $M_0 > 2$, such that given any $M > M_0$ there is a clopen partition $\mathcal{P} = \{\varphi^w C_k : w \in Z_k, 1 \leq k \leq K\}$ of X with the following properties.*

   (1) $x_0 \in \cup_{k=1}^{K} C_k \subset C$,
   (2) for all k, Z_k contains all vectors $v \in \mathbb{Z}^d$ with $\|v\| \leq M/2$,

(3) *for all k, Z_k contains no vectors $v \in \mathbb{Z}^d$ with $\|v\| \geq 2M$,*
(4) *for any $v \in \mathbb{Z}^d$, $|Z_k \,\triangle\, (Z_k - v)| < L \|v\| M^{d-1}$.*

Proof. Given C as in the hypothesis, by Proposition 3.6, there is an M_0 such that for $M > M_0$ there is an M-regular clopen set D containing x_0. Let $P_1, P_2, \ldots, P_K$ be the prototiles appearing in $\tau(R_D(x))$ for $x \in X$, and $C_k = \{x \in C : P(x) = P_k\}$. Via the procedure described after Proposition 3.6, we obtain clopen partition $\mathcal{P} = \{\varphi^w C_k : w \in Z_k, 1 \leq k \leq K\}$ of X. Note that any difference between $P_k \cap \mathbb{Z}^d$ and Z_k takes place only if we have an intersection of two sets $\varphi^w C_k$ and $\varphi^v C_l$ in the original cover and in this case $\|w\| = \|v\| > M/2$. Thus the intersection of the original P_k with the ball around the origin of radius $M/2$ is not affected, and it follows that Z_k satisfy properties 2 and 3. By Lemmas 3.2, the collection of vectors in $Z_k \,\triangle\, (Z_k - v)$ is the union of at most b sets, each of which is a collection of $\mathbb{Z}^d$-vectors within distance $\|v\|$ of a subset of a $(d-1)$-dimensional hyperplane in $\mathbb{R}^d$. Because these hyperplane subsets are within the tile P_k, they have diameter less than $4M$. The bound in 4 follows. $\square$

We refer to a paritititon $\mathcal{P} = \{\varphi^w C_k : w \in Z_k, 1 \leq k \leq K\}$ with the properties described above as a *Voronoi-Rohlin partition centered at C*. In the next section we will use the notation P_k in place of Z_k for the integer partitions.

3.2. **The Ordered Group** $\left(G\left(\varphi\right), G\left(\varphi\right)_+\right)$. Given $\left(X, \varphi, \mathbb{Z}^d\right)$ a minimal free Cantor system, consider the group $C(X, \mathbb{Z})$ under addition, generated by indicator functions 1_A of clopen sets $A \subset X$. Let $B(\varphi) \subset C(X, \mathbb{Z})$ denote the set of φ-*coboundaries*, i.e., functions which are sums of functions of the form $1_A - 1_{\varphi^v A}$ where A is clopen and $v \in \mathbb{Z}^d$.

Definition 3.8. *We set:*

$$
\begin{aligned}
G(\varphi) &= C(X, \mathbb{Z}) / B(\varphi) \\
G(\varphi)_+ &= \{[f] : f(x) \geq 0 \text{ for all } x \in X\}
\end{aligned}
$$

The pair $\left(G(\varphi), G(\varphi)_+\right)$ is an *ordered group,* that is, $G(\varphi)$ is a countable abelian group, and $G(\varphi)_+$ is a subset of $G(\varphi)$ satisfying the following [4].

(1) $G(\varphi)_+ + G(\varphi)_+ \subset G(\varphi)_+$
(2) $G(\varphi)_+ + \left(-G(\varphi)_+\right) = G(\varphi)$
(3) $G(\varphi)_+ \cap \left(-G(\varphi)_+\right) = 0$

Notation 3.9. *For g_1 and g_2 in an ordered group (G, G_+), we will use the notation*

- *$g_2 \geq g_1$ if $g_2 - g_1 \in G_+$*
- *$g_2 > g_1$ if $g_2 \geq g_1$ and $g_2 \neq g_1$.*

We need to employ Voronoi-Rohlin partitions in order to make use of the ordered group $\left(G(\varphi), G(\varphi)_+\right)$ as an invariant for bounded orbit injection equivalence. The next two are the main lemmas along these lines.

Lemma 3.10. *Let $\left(X, \varphi, \mathbb{Z}^d\right)$ be a minimal Cantor system, and let $f, g \in C(X, \mathbb{Z})$. Suppose $[f] < [g]$ in $\left(G(\varphi), G(\varphi)_+\right)$. Then there are constants $c > 0$ and $M_0 > 2$ such that if $M > M_0$, C is M-regular and $\mathcal{P} = \{\varphi^w C_k : w \in P_k, 1 \leq k \leq K\}$ is a Voronoi-Rohlin partition centered at C then for $1 \leq k \leq K$ and $x \in C_k$,*

$$
\sum_{v \in P_k} g\varphi^v(x) - \sum_{v \in P_k} f\varphi^v(x) > cM^d
$$

Proof. We have that $g - f = h + \sum_{i=1}^{I} \left(1_{A_i} - 1_{\varphi^{v(i)} A_i}\right)$ where $h(x) \geq 0$ for all $x \in X$, and the A_i are clopen. Since $[f] < [g]$ there is a clopen set U upon which $h(x) > 0$. If C is M-separated then each P_k contains all integer lattice points which are within $M/2$ of the origin. Because U is r-syndetic for some $r > 0$, we know that there is a constant ℓ such that if M is sufficiently large and C is any M-separated set, then for all $x \in X$,

$$\sum_{v \in P_k} h\varphi^v(x) > \ell M^d$$

On the other hand for each i,

$$\left| \sum_{v \in P_k} \left(1_{A_i} - 1_{\varphi^{v(i)} A_i}\right)(\varphi^v(x)) \right| = \left| \sum_{v \in P_k} 1_{A_i}(\varphi^v(x)) - 1_{A_i}\left(\varphi^{v-v(i)}(x)\right) \right|$$
$$\leq |P_k \triangle (P_k - v(i))|$$

By Theorem 3.7, we have

$$|P_k \triangle (P_k - v(i))| \leq LM^{d-1} \|v(i)\|$$

If M_0 is large enough to insure $c = \ell - \frac{1}{M} L \sum_{i=1}^{I} \|v(i)\| > 0$, then

$$\sum_{v \in P_k} g\varphi^v(x) - \sum_{v \in P_k} f\varphi^v(x) = \sum_{v \in P_k} h\varphi^v(x) + \sum_{v \in P_k} \left(1_{A_i} - 1_{\varphi^{v(i)} A_i}\right)(\varphi^v(x))$$
$$> \ell M^d - \sum_{i=1}^{I} LM^{d-1} \|v(i)\|$$
$$= cM^d$$

$\square$

Lemma 3.11. *Let $\left(X, \varphi, \mathbb{Z}^d\right)$ be a minimal Cantor system, and let $f, g \in C(X, \mathbb{Z})$. Suppose $\mathcal{P} = \{\varphi^w C_k : w \in P_k, 1 \leq k \leq K\}$ is a Voronoi-Rohlin partition centered at C such that for all $x \in C_k$,*

$$\sum_{v \in P_k} g\varphi^v(x) \geq \sum_{v \in P_k} f\varphi^v(x)$$

then $[g] \geq [f]$. Moreover, if the above inequality above holds for all $x \in C_k$, and for some k and $x \in C_k$ the inequality is strict then $[g] > [f]$.

Proof. Suppose $\mathcal{P}$ is a Voronoi-Rohlin partition satisfying the above properties. For any function $h \in C(X, \mathbb{Z})$, let $\widehat{h} \in C(X, \mathbb{Z})$ be the following function

$$\widehat{h}(x) = \sum_{k=1}^{K} \sum_{w \in P_k} 1_{C_k}(x) \cdot h\varphi^w(x)$$

Now

$$h - \widehat{h} = \sum_{k=1}^{K} \sum_{w \in P_k} 1_{\varphi^w C_k} \cdot h - \sum_{k=1}^{K} \sum_{w \in P_k} 1_{C_k} \cdot h\varphi^w$$
$$= \sum_{k=1}^{K} \sum_{w \in P_k} \left(1_{C_k} \varphi^{-w} \cdot h - 1_{C_k} \cdot h\varphi^w\right)$$

So $h - \widehat{h} \in B(\varphi)$ for any function $h \in C(X, \mathbb{Z})$.

The assumptions assert that $\widehat{g} - \widehat{f}$ is a nonnegative function, which means that $[g] \geq [f]$. If in addition, $\widehat{g} - \widehat{f}$ evaluates to a positive value at one $x \in X$, then $[g] - [f] \neq 0$, which implies $[g] > [f]$. $\hfill\square$

In what follows we will use the following properties of $\left(G\left(\varphi\right), G\left(\varphi\right)_+\right)$

(1) $\left(G\left(\varphi\right), G\left(\varphi\right)_+\right)$ is *weakly unperforated*, i.e., if $n\left[f\right] > 0$ with $n \in \mathbb{Z}$ and $[f] \in G\left(\varphi\right)$ then $[f] > 0$.
(2) $\left(G\left(\varphi\right), G\left(\varphi\right)_+\right)$ has the strong *Reisz property*, i.e., if $[g_1]$, $[g_2]$, $[f_1]$, $[f_2]$ are in $G\left(\varphi\right)$ with $[f_j] > [g_i]$ for all $i, j \in \{1, 2\}$ then there exists an $[h] \in G$ such that $[f_j] > [h] > [g_i]$ for all $i, j \in \{1, 2\}$.
(3) $\left(G\left(\varphi\right), G\left(\varphi\right)_+\right)$ is *simple*, i.e., if for every $[f] \in G\left(\varphi\right)_+ \backslash \{0\}$ and $[g] \in G\left(\varphi\right)$, there is an $n \in \mathbb{N}$ such that $n\left[f\right] > [g]$.

These properties were proven in [4], along with the claim that $G\left(\varphi\right)$ is torsion-free. The torsion-freeness part of the proof was later shown to be incorrect, leading to some interesting developments, see [8]. Nevertheless, it is true that the above properties hold for $\left(G\left(\varphi\right), G\left(\varphi\right)_+\right)$; for example, they follow from Lemma 3.11.

3.3. **The ordered group as an invariant.** In the following, we use Voronoi-Rohlin partitions to prove that if: $\left(X, \varphi, \mathbb{Z}^d\right)$ and $\left(Y, \psi, \mathbb{Z}^d\right)$ are bounded orbit injection equivalent then:

$$\left(G\left(\varphi\right), G\left(\varphi\right)_+\right) \cong \left(G\left(\psi\right), G\left(\psi\right)_+\right).$$

Theorem 3.12. *Let* $d \geq 1$, *and suppose* $\left(X, \varphi, \mathbb{Z}^d\right)$ *and* $\left(Y, \psi, \mathbb{Z}^d\right)$ *are bounded orbit injection equivalent. Then* $\left(G\left(\varphi\right), G\left(\varphi\right)_+\right) \cong \left(G\left(\psi\right), G\left(\psi\right)_+\right)$.

Proof. It suffices to consider the case where there is a bounded orbit injection $\theta : X \to Y$. Consider the homomorphism $h_\theta : C(X, \mathbb{Z}) \to G(\psi)$ defined on generators by $h_\theta\left(1_A\right) = [1_{\theta A}]$ for any clopen set $A \subset X$.

To show that h_θ gives a well-defined homomorphism on $G\left(\varphi\right)$, we wish to show that $h_\theta\left(1_A - 1_{\varphi^v A}\right) = [0]$ for any clopen set $A \subset X$ and $v \in \mathbb{Z}^d$. A clopen set A partitions into finitely many clopen sets A_i such that for each i, there is a $v(i)$ with $\theta\varphi^v(x) = \psi^{v(i)}\theta(x)$ for all $x \in A_i$. Therefore,

$$
\begin{aligned}
h\left(1_A - 1_{\varphi^v A}\right) &= \left[1_{\theta A} - 1_{\theta\varphi^v A}\right] \\
&= \sum_i \left[1_{\theta A_i} - 1_{\theta\varphi^v A_i}\right] \\
&= \sum_i \left[1_{\theta A_i} - 1_{\psi^{v(i)}\theta A_i}\right] \\
&= 0
\end{aligned}
$$

Now we will construct an inverse homomorphism. Apply Proposition 3.6 to $\theta(X)$ to obtain an M-regular subset C of $\theta(X)$ with $M > 2$, and an associated Voronoi-Rohlin partition centered at C, $\mathcal{P} = \{\psi^w C_k : w \in P_k\}$. We can assume, after partitioning the clopen sets C_k further if necessary, that each partition element $\psi^w C_k$ is a subset of $\theta(X)$ or $\theta(X)^c$. From this, the set $\theta(X)^c$ is a disjoint union of clopen sets of the form $\psi^w C_k$, i.e.,

$$\theta\left(X\right)^c = \cup_{k=1}^{K} \cup_{j=1}^{J(k)} \psi^{w(j,k)} C_k$$

where $w(j,k) \in P_k$ for all (j,k). Let $f \in C(Y, \mathbb{Z})$. Then let $\widehat{f} \in C(Y, \mathbb{Z})$ be the following function

$$\widehat{f}(x) = \begin{cases} 0 & \text{if } x \in Y \setminus \theta(X) \\ f(x) & \text{if } x \in \theta(X) \setminus C \\ f(x) + \sum_{k=1}^{K} \sum_{j=1}^{J(k)} 1_{C_k}(x) \cdot f\psi^{w(j,k)}(x) & \text{if } x \in C \end{cases}$$

It is not difficult to see that $f \mapsto \widehat{f}$ is a homomorphism, and therefore the map $g_\theta : C(Y, \mathbb{Z}) \to G(\varphi)$ defined by $g_\theta : f \mapsto \left[\widehat{f}\theta\right]$ is a homomorphism. We wish to see that g_θ applied to a ψ-coboundary is equal to $[0]$ in $G(\varphi)$. Recall that any ψ-coboundary is equal to the sum of functions of the form $1_{B_i} - 1_{\psi^{v(i)}B_i}$ where the sets $B_i \subset Y$ are clopen and vectors $v(i) \in \mathbb{Z}^d$; note that the B_i need not be distinct nor disjoint. Further, by subdividing the sets if necessary, we may assume that each set B_i and $\psi^{v(i)}B_i$ is a subset of an element of the Voronoi-Rohlin partition $\mathcal{P} = \{\psi^w C_k : w \in P_k, 1 \le k \le K\}$. Fix i and set $B = B_i$, $v = v(i)$. Then $B \subset \psi^w C_k$ for some w, k. If $B \subset \theta(X)$, then $\widehat{1}_B = 1_B$ since $1_B \equiv 0$ on $Y \setminus \theta(X)$. If $B \subset Y \setminus \theta(X)$, then $\widehat{1}_B = 1_{C_k} \cdot 1_B \circ \psi^w = 1_{\psi^{-w}B}$. In either case there is a vector u such that $\widehat{1}_B = 1_{\psi^u B}$ and $\psi^u B \subset \theta(X)$. A similar fact is true of $\psi^v B$, there is a vector t such that $\widehat{1}_{\psi^v B} = 1_{\psi^t B}$ with $\psi^t B \subset \theta(X)$. Setting $\widehat{B} = \psi^u B$, and $\widehat{v} = t - u$ we have $\widehat{1}_B - \widehat{1}_{\psi^v B} = 1_{\widehat{B}} - 1_{\psi^{\widehat{v}}\widehat{B}}$.

Now let us consider $1_{\widehat{B}}\theta - 1_{\psi^{\widehat{v}}\widehat{B}}\theta = 1_{\theta^{-1}\widehat{B}} - 1_{\theta^{-1}\psi^{\widehat{v}}\widehat{B}}$. Because $\widehat{B}$ and $\psi^{\widehat{v}}\widehat{B}$ are subsets of $\theta(X)$, for each $x \in \theta^{-1}\widehat{B}$ there is a vector $\alpha(x)$ such that $\theta\varphi^{\alpha(x)}(x) = \psi^{\widehat{v}}\theta(x)$. Further the function α is continuous and takes on finitely many values $u(1), u(2), \ldots, u(I)$. Therefore, $1_{\theta^{-1}\widehat{B}} - 1_{\theta^{-1}\psi^{\widehat{v}}\widehat{B}}$ is a finite sum of the form $1_{A(i)} - 1_{\varphi^{u(i)}A(i)}$ where $A(i)$ is the clopen set $A(i) = \{x : \alpha(x) = u(i)\}$. It follows that g_θ applied to a ψ-coboundary is a φ-coboundary. In particular this means that $g_\theta : G(\psi) \to G(\varphi)$ is well-defined.

Now consider $g_\theta h_\theta$ applied to a function 1_A where $A \subset X$ is clopen. Then since $\theta(A) \subset \theta(X)$, $\widehat{1}_{\theta A} = 1_{\theta A}$. Further since θ is injective, $\theta^{-1}\theta A = A$. Thus,

$$\begin{aligned} g_\theta h_\theta [1_A] &= g_\theta [1_{\theta A}] \\ &= \left[\widehat{1}_{\theta A}\theta\right] \\ &= [1_{\theta A}\theta] \\ &= [1_{\theta^{-1}\theta A}] \\ &= [1_A] \end{aligned}$$

Now consider $h_\theta g_\theta$ applied to a function 1_B where $B \subset Y$ is clopen and $B \subset \psi^w C_k$ for some w, k. Then recall that $\widehat{1}_B = 1_{\psi^u B}$ for some u with $\psi^u B \subset \theta(X)$

$$\begin{aligned} h_\theta g_\theta [1_B] &= h_\theta \left[\widehat{1}_B\theta\right] \\ &= h_\theta [1_{\psi^u B}\theta] \\ &= h_\theta [1_{\theta^{-1}\psi^u B}] \\ &= [1_{\theta\theta^{-1}\psi^u B}] \\ &= [1_{\psi^u B}] \\ &= [1_B] \end{aligned}$$

Therefore, $g_\theta h_\theta$ and $h_\theta g_\theta$ are both identity maps and $G(\varphi)$, $G(\psi)$ are isomorphic.

To see that the positive cones are preserved, consider $f \in C(X, \mathbb{Z})$ with $f(x) \geq 0$ for all $x \in X$. Then $f = \sum c_i 1_{A_i}$ where $c_i > 0$, A_i are clopen. $h_\theta[f] = [\sum c_i 1_{\theta A_i}] \in G(\psi)_+$.

Conversely, suppose $f \in C(Y, \mathbb{Z})$ and $f(y) \geq 0$ for all $y \in Y$. Then $\widehat{f}(y) \geq 0$ for all $y \in Y$, and $\widehat{f}\theta(x) \geq 0$ for all $x \in X$. Thus $g_\theta[f] \in G(\varphi)_+$.   □

Suppose $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ are minimal Cantor systems which are bounded orbit injection equivalent by virtue of bounded orbit injections into a common system $(Z, \alpha, \mathbb{Z}^d)$. Below we give a condition on the isomorphisms created in Theorem 3.12 which guarantees that there is a bounded orbit injection from $(X, \varphi, \mathbb{Z}^d)$ into $(Y, \psi, \mathbb{Z}^d)$. This in turn, leads to a proof that bounded orbit injection equivalence is in fact an equivalence relation for $d > 2$ (the cases where $d = 1, 2$ are already covered by [7]).

Theorem 3.13. *Suppose there exist bounded orbit injections θ_1 and θ_2 from systems $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ into $(Z, \alpha, \mathbb{Z}^d)$, and that $h_{\theta_1}[1_X] < h_{\theta_2}[1_Y]$. Then there is a bounded orbit injection from $(X, \varphi, \mathbb{Z}^d)$ into $(Y, \psi, \mathbb{Z}^d)$.*

Proof. By Lemma 3.10, there is a Voronoi-Rohlin partition:
$$\mathcal{P} = \{\alpha^v C_k : w \in P_k, 1 \leq k \leq K\}$$
of Z centered at $C = \cup_{k=1}^K C_k$ such that for $x \in C_k$,
$$\sum_{v \in P_k} 1_{\theta_2(Y)} \alpha^v(x) > \sum_{v \in P_k} 1_{\theta_1(X)} \alpha^v(x)$$
By further refining if necessary, we may assume $\mathcal{P}$ refines both $\{\theta_1(X), Z \setminus \theta_1(X)\}$ and $\{\theta_2(Y), Z \setminus \theta_2(Y)\}$. Now for each k, we define an injection
$$\rho_k : \{v \in P_k : \alpha^v C_k \subset \theta_1(X)\} \to \{v \in P_k : \alpha^v C_k \subset \theta_2(Y)\}$$
For $x \in \alpha^v C_k \cap \theta_1(X) \in \mathcal{P}$ set $\pi(x) = \alpha^{\rho_k(v)-v}(x)$. Then $\pi(\theta_1(X)) \subset \theta_2(Y)$ and we have a bounded orbit injection from $(X, \varphi, \mathbb{Z}^d)$ into $(Y, \psi, \mathbb{Z}^d)$ defined by $\theta_2^{-1}\pi\theta_1$.   □

Let $m > 0$. By the *tower of height m over* $(X, \varphi, \mathbb{Z}^d)$, we mean the system of the form $(X(m), \widehat{\varphi}, \mathbb{Z}^d)$ where $X(m) = X \times \{0, 1, \ldots, m-1\}^d$ and $\widehat{\varphi}$ is defined by the following. Let $x \in X$, $v \in \mathbb{Z}^d$ and $u \in \{0, 1, \ldots, m-1\}^d$. Then $v + u$ can be written in the form $mw + r$ where $w \in \mathbb{Z}^d$ and $r \in \{0, 1, \ldots, m-1\}^d$. We then define $\widehat{\varphi}^v(x, u) = (\varphi^w(x), r)$. Note that $X \times \{0, 1, \ldots, m-1\}^d$ is a Cantor set, and that if φ is a minimal free $\mathbb{Z}^d$-action of X, then $\widehat{\varphi}$ is a minimal free $\mathbb{Z}^d$-action of $X(m)$. Further note that there is a bounded orbit injection from $(X, \varphi, \mathbb{Z}^d)$ into a tower of height m over $(X, \varphi, \mathbb{Z}^d)$ given by $\theta(x) = (x, 0)$. In this, the isomorphism $h_\theta : G(\varphi) \to G(\widehat{\varphi})$ satisfies $m^d h_\theta[1_X] = [1_{X(m)}]$.

Lemma 3.14. *Suppose there exist bounded orbit injections θ_1 and θ_2 from systems $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ into $(Z, \alpha, \mathbb{Z}^d)$. Then for some $m > 0$ there is a bounded orbit injection from $(X, \varphi, \mathbb{Z}^d)$ into a tower of height m over $(Y, \psi, \mathbb{Z}^d)$.*

Proof. By the fact that $G(\alpha)$ is simple, there is an $n > 0$ such that $h_{\theta_1}[1_X] < n h_{\theta_2}[1_Y]$. Fix $m > 0$ so that $m^d > n$. Notice that there is a bounded orbit injection $\widehat{\theta}_1$ from $(X, \varphi, \mathbb{Z}^d)$ into a tower of height m over $(Z, \alpha, \mathbb{Z}^d)$ defined by

$\widehat{\theta}_1(x) = (\theta_1(x), 0)$. Also notice that there is a bounded orbit injection $\widehat{\theta}_2$ from the tower of height m over $(Y, \psi, \mathbb{Z}^d)$ into the tower of height m over $(Z, \alpha, \mathbb{Z}^d)$, defined by $\widehat{\theta}_2(y, u) = (\theta_2(y), u)$ for $u \in \{0, 1, \ldots, m-1\}^d$. Now

$$h_{\widehat{\theta}_1}[1_X] < n h_{\widehat{\theta}_2}[1_Y] < m^d h_{\widehat{\theta}_2}[1_Y] = h_{\widehat{\theta}_2}\left[1_{Y(m)}\right]$$

Therefore, by Lemma 3.13, there is a bounded orbit injection from $(X, \varphi, \mathbb{Z}^d)$ into $\left(Y(m), \widehat{\psi}, \mathbb{Z}^d\right)$. $\qquad \square$

Theorem 3.15. *Bounded orbit injection equivalence is an equivalence relation.*

Proof. Reflexivity and symmetry are clear, we prove transitivity. Suppose there are bounded orbit injections from $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ into a common system and there are bounded orbit injections from $(Y, \psi, \mathbb{Z}^d)$ and $(Z, \alpha, \mathbb{Z}^d)$ into a common system. Then by Lemma 3.14, there is an $m > 0$ such that there exist bounded orbit injections from both $(X, \varphi, \mathbb{Z}^d)$ and $(Z, \alpha, \mathbb{Z}^d)$ into a tower of height m over $(Y, \psi, \mathbb{Z}^d)$. $\qquad \square$

3.4. **Orbit equivalences from injections.** Now let us suppose that two minimal Cantor systems $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ are bounded orbit injection equivalent, with bounded orbit injections θ_1 and θ_2 into a common system $(Z, \alpha, \mathbb{Z}^d)$. Let h_{θ_1} and h_{θ_2} be the isomorphisms as in Theorem 3.12. We show that if $h_{\theta_1}[1_X] = h_{\theta_2}[1_Y]$ holds then there is a *bounded orbit equivalence* θ from $(X, \varphi, \mathbb{Z}^d)$ to $(Y, \psi, \mathbb{Z}^d)$, i.e., an orbit equivalence θ in which the function $\eta : X \times \mathbb{Z}^d \to \mathbb{Z}^d$ satisfying $\theta(\varphi^v(x)) = \psi^{\eta(x,v)}\theta(x)$ is continuous.

Let $(X, \varphi, \mathbb{Z}^d)$ be a minimal Cantor system. By the *full group of* φ we mean the collection of homeomorphisms $\pi : X \to X$ such that for each x, $\pi(x) = \varphi^{\zeta(x)}(x)$ for some $\zeta(x) \in \mathbb{Z}^d$. By the *topological full group of* φ we mean the collection of full group elements that have the property that the associated cocycle function $\zeta : X \to \mathbb{Z}^d$ is continuous.

Theorem 3.16. *Let $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ be two minimal Cantor systems which are bounded orbit injection equivalent, with bounded orbit injections θ_1 and θ_2 into a common system $(Z, \alpha, \mathbb{Z}^d)$. Let $h_{\theta_1} : (G(\varphi), G(\varphi)_+) \to (G(\alpha), G(\alpha)_+)$ and $h_{\theta_2} : (G(\psi), G(\psi)_+) \to (G(\alpha), G(\alpha)_+)$ be the isomorphisms as in Theorem 3.12. Suppose $h_{\theta_1}[1_X] = h_{\theta_2}[1_Y]$ in $G(\alpha)$. Then there is a bounded orbit equivalence θ from $(X, \varphi, \mathbb{Z}^d)$ to $(Y, \psi, \mathbb{Z}^d)$. Further, $h_\theta = h_{\theta_2}^{-1} h_{\theta_1}$ is an isomorphism from $(G(\varphi), G(\varphi)_+)$ to $(G(\psi), G(\psi)_+)$.*

Proof. First consider the case where $\theta_1(X) = Z$. Then $\theta_2(Y) = Z$ as well, for otherwise $Z \setminus \theta_2(Y)$ is a nonempty clopen set and $h_{\theta_1}[1_X] = [1_Z] = h_{\theta_2}[1_Y] + [1_{Z \setminus \theta_2(Y)}] > h_{\theta_2}[1_Y]$. But if $\theta_1(X) = Z = \theta_2(Y)$ then we can simply let $\theta = \theta_2^{-1}\theta_1$, and we are done.

Assume then that $[1_Z] > [1_{\theta_1(X)}] = [1_{\theta_2(X)}]$. Then

$$1_{\theta_1(X)} - 1_{\theta_2(Y)} = \sum_{i=1}^{I}\left(1_{A_i} - 1_{\alpha^{v(i)}A_i}\right)$$

where $A_i \subset Z$ are clopen and $v(i) \in \mathbb{Z}^d$. Via Lemma 3.10, there is a $c > 0$ and $M_0 > 2$ such that if C is M-regular and $\mathcal{P} = \{\varphi^w C_k : w \in P_k, 1 \le k \le K\}$ is a

Voronoi-Rohlin partition centered at C then for $1 \leq k \leq K$ and $x \in C_k$, both the following hold

$$\sum_{v \in P_k} 1_{\theta_2(Y)} \alpha^v(x) > cM^d$$

$$\sum_{v \in P_k} 1_Z \alpha^v(x) - \sum_{v \in P_k} 1_{\theta_2(Y)} \alpha^v(x) > cM^d$$

Choose M such that $cM > L \sum_{i=1}^I \|v(i)\|$ where L is the constant from Theorem 3.7 (item 4).

Let $\mathcal{P} = \{\alpha^w C_k : w \in P_k, 1 \leq k \leq K\}$ be a Voronoi-Rohlin partition of Z centered at an M-regular clopen set C. If necessary, partition the clopen sets C_k so that $\mathcal{P}$ refines $\{A_i, Z \setminus A_i\}$ and $\{\alpha^{v(i)} A_i, Z \setminus \alpha^{v(i)} A_i\}$ for each i.

Then for each i,

$$A_i = \cup_{k=1}^K \cup_{j=1}^{J(k)} \alpha^{w(i,j,k)} C_k$$

where $w(i,j,k) \in P_k$ for all (i,j,k). Let $B_i \subset A_i$ be the union of sets $\alpha^{w(i,j,k)} C_k$ over the indices (i,j,k) where $w(i,j,k) + v(i) \notin P_k$. The number of such indices (i,j,k) is bounded above by

$$\sum_{i=1}^I |P_k \triangle (P_k - v(i))| < \sum_{i=1}^I L \|v(i)\| M^{d-1}$$
$$< cM^d$$

For each $x \in B_i$, $x \in \alpha^{w(x)} C_{k(x)}$ for some $1 \leq l(x) \leq K$ and $w(x) \in P_{k(x)}$ and $\alpha^{v(i)}(x)$ is in $\alpha^{u(x)} C_{l(x)}$ for some $1 \leq l(x) \leq K$ for some $u(x) \in P_{l(x)}$. The map $x \mapsto (k(x), w(x), l(x), u(x))$ is continuous.

For each pair of indices k and w with $\alpha^w C_k \subset B_i$, we select a vector $r(i,w,k) \in P_k$ so that $\alpha^{r(i,w,k)} C_k \subset Z \setminus \theta_2(Y)$. We do so in such a way that if $(i,w) \neq (i',w')$ then $r(i,w,k) \neq r(i',w',k)$. This is possible because

$$\#\{(i,k,w) : \alpha^w C_k \subset B_i \text{ for some } i\} = \sum_{i=1}^I |P_k \triangle (P_k - v(i))|$$
$$< cM^d$$
$$< \sum_{v \in P_k} 1_{Z \setminus \theta_2(Y)} \alpha^v(x)$$

By the same reasoning, for each index l and u with $\alpha^u C_l \subset \alpha^{v(i)} B_i$, we select a vector $s(i,u,l) \in P_l$ so that $\alpha^{s(i,u,l)} C_l \subset \theta_2(Y)$, and that if $(i,u) \neq (i',u')$ then $s(i,u,l) \neq s(i',u',l)$.

Define $\pi : \theta_2(Y) \to Z$, an element of the topological full group of α as follows. Suppose $y = \alpha^{v(i)}(x)$ for some $x \in B_i$ where $B_i \subset \alpha^w C_k \in \mathcal{P}$, and $y \in \alpha^u C_l \in \mathcal{P}$. Define $\pi(\alpha^{s(i,u,l)-u} y) = \alpha^{r(i,w,k)-w}(x)$. We set $\pi \equiv id$ on the complement of $\cup_{i,u,l} \alpha^{s(i,u,l)-u} C_l$.

Now replace the bounded orbit injection $\theta_2 : Y \to Z$ with the bounded orbit injection $\pi\theta_2 : Y \to Z$.

Fix k, and let us consider $z \in C_k$ and the sum

$$\sum_{v \in P_k} \left(1_{\theta_1(X)}(\alpha^v z) - 1_{\pi\theta_2(Y)}(\alpha^v z)\right) = \sum_{v \in P_k} \left(1_{\theta_1(X)}(\alpha^v z) - 1_{\theta_2(Y)}(\alpha^v z)\right)$$
$$+ \sum_{v \in P_k} \left(1_{\theta_2(Y)}(\alpha^v z) - 1_{\pi\theta_2(Y)}(\alpha^v z)\right)$$

We rewrite the first and second terms in the sum as follows.

$$\sum_{v \in P_k} \left(1_{\theta_1(X)}(\alpha^v z) - 1_{\theta_2(Y)}(\alpha^v z)\right) = \sum_{v \in P_k}\sum_i \left(1_{B_i}(\alpha^v z) - 1_{\alpha^{v(i)}B_i}(\alpha^v z)\right)$$
$$= \sum_i \#\left\{v \in P_k : \alpha^v C_k \subset B_i\right\}$$
$$- \sum_i \#\left\{v \in P_k : \alpha^{v - v(i)}C_k \subset B_i\right\}$$

$$\sum_{v \in P_k} \left(1_{\theta_2(Y)}(\alpha^v z) - 1_{\pi\theta_2(Y)}(\alpha^v z)\right) =$$
$$\sum_i \#\left\{v \in P_k : v = s(i, u, k) \text{ for some } u \in P_k\right\}$$
$$- \sum_i \#\left\{v \in P_k : r(i, w, k) = v \text{ for some } w \in P_k\right\}$$

Fixing i, these sums cancel. This means that for each $z \in C_k$, there is a one-to-one correspondence $\zeta : \{v \in P_k : \alpha^v(z) \subset \theta_1(X)\} \to \{v \in P_k : \alpha^v(z) \subset \pi\theta_2(Y)\}$.

With this, we can set up a bounded orbit equivalence h from X to Y by taking $h(x) = \theta_2^{-1}\pi^{-1}\alpha^{\zeta(v) - v}\theta_1(x)$ where $\theta_1(x) \in \alpha^v C_k$. $\qquad\square$

Finally we examine the situation where two minimal Cantor systems $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ are bounded orbit injection equivalent, with bounded orbit injections θ_1 and θ_2 into a common system $(Z, \alpha, \mathbb{Z}^d)$ with the property that $h_{\theta_1}[1_X] - h_{\theta_2}[1_Y]$ is an infinitesimal, i.e., an element of the subgroup $Inf(G(\alpha))$ as defined below.

Definition 3.17. *For a simple ordered group (G, G_+) we define $Inf(G)$ to be the following subgroup*

$$Inf(G) = \{g \in G : ng < h \text{ for any } n \in \mathbb{Z}, \text{ and any } h \in G_+ \setminus \{0\}\}$$

The type of bounded orbit injection equivalence discussed here corresponds to a kind of topological version of the notion of *even* Kakutani equivalence in measure theoretic dynamics (see for example, [2]). It follows from the main results of [14] that $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ are orbit equivalent. We obtain a stronger form of orbit equivalence in this setting, not bounded, but where for any $v \in \mathbb{Z}^d$ the cocycle $\eta(\cdot, v)$ is continuous except at two points $\{x_0, \varphi^{-v}x_0\}$.

We first show over the next three propositions that if indicator functions of clopen sets differ by a infinitesimal then there is a full group element mapping one to the other. The proof is essentially an adaptation of an argument in [5] to the case of $\mathbb{Z}^d$-actions.

Proposition 3.18. *Let $(X, \varphi, \mathbb{Z}^d)$ be a minimal Cantor system. Let A, B be two clopen sets in X such that $[1_A] - [1_B] \in Inf(\varphi)$. Then given a proper clopen subset $D \subset A$ and point $b \in B$, there is a clopen subset $E \subset B \setminus \{b\}$ such that $[1_D] < [1_E]$.*

Proof. Let $\varepsilon = [1_A] - [1_B] \in Inf(\varphi)$. Since $D \subset A$ is a proper subset, $A \setminus D$ is a nonempty clopen set and $[1_{A\setminus D}] > 0$. Now $[1_B] + \varepsilon - [1_D] = [1_A] - [1_D] = [1_{A\setminus D}] > 0$. Because $(G(\varphi), G(\varphi)_+)$ is simple, there exists an $n \in \mathbb{N}$ such that $n([1_B] + \varepsilon - [1_D]) - [1] > 0$. Because ε is infinitesimal, $[1] - n\varepsilon > 0$, adding this to $n([1_B] + \varepsilon - [1_D]) - [1]$, we obtain $n([1_B] - [1_D]) > 0$. Because the ordered group is weakly unperforated, $[1_B] - [1_D] > 0$.

Now use Lemma 3.10 to find a M_0 such that whenever C is M-regular for $M > M_0$ and $\mathcal{P} = \{\varphi^w C_k : w \in P_k, 1 \leq k \leq K\}$ is a Voronoi-Rohlin partition of X centered at C then for all $x \in C_k$,

$$\sum_{v \in P_k} 1_D \varphi^v(x) - \sum_{v \in P_k} 1_B \varphi^v(x) > 2$$

Apply Proposition 3.6 to construct an M-regular clopen set $C \subset B$ containing b with $M > M_0$ and the corresponding Voronoi-Rohlin partition

$$\mathcal{P} = \{\varphi^w C_k : w \in P_k, 1 \leq k \leq K\}.$$

By partitioning C_k if necessary, we may assume that $\mathcal{P}$ is a finer partition than both $\{D, X \setminus D\}$ and $\{B, X \setminus B\}$. One of these partition elements B is a union of the form $\cup_{k=1}^{K} \cup_{j=1}^{J(k)} \varphi^{w(j,k)} C_k$ and one of these sets, say $\varphi^{w(1,1)} C_1$, contains the point b. Let E be the clopen set $B \setminus \varphi^{w(1,1)} C_1$. Then $E \subset B$, $b \notin E$, and

$$\sum_{v \in P_k} 1_D \varphi^v(x) < \sum_{v \in P_k} 1_E \varphi^v(x).$$

Therefore, $[1_E] - [1_D] \in G(\varphi)_+ \setminus [0]$ which gives the result. $\square$

Lemma 3.19. *Let $(X, \varphi, \mathbb{Z}^d)$ be a minimal Cantor system. Let A, B be two clopen sets in X with $[1_A] - [1_B] \in Inf(\varphi)$. Fix $x_0 \in A$, $y_0 \in B$ and let $\epsilon > 0$ be given. Then there is an element π of the topological full group of φ and a clopen set $A' \subset A \setminus \{x_0\}$ such that*

(1) $A' \supset A \setminus B(x_0, \epsilon)$
(2) $\pi(A') \subset B \setminus \{y_0\}$
(3) $\pi^2 = id$
(4) $\pi|_{X \setminus A'} = id$

Proof. Without loss of generality, A and B are disjoint, otherwise set $\pi = id$ on $A \cap B$. Let A' be any clopen set such that $A \setminus B(x_0, \epsilon) \subset A' \subset A \setminus \{x_0\}$. Then by the previous proposition there is a clopen set $B' \subset B \setminus \{y_0\}$ such that $[1_{B'}] > [1_{A'}]$. Now use Lemma 3.10 to find a M_0 such that whenever C is M-regular for $M > M_0$ and $\mathcal{P} = \{\varphi^w C_k : w \in P_k, 1 \leq k \leq K\}$ is a Voronoi-Rohlin partition of X centered at C then for all $x \in C$,

$$\sum_{v \in P_k} 1_{A'} \varphi^v(x) < \sum_{v \in P_k} 1_{B'} \varphi^v(x)$$

Next apply Proposition 3.6 to construct an M-regular clopen set $C \subset A$ containing x_0 with $M > M_0$, and the corresponding Voronoi-Rohlin partition $\mathcal{P} = \{\varphi^w C_k : w \in P_k, 1 \leq k \leq K\}$. By partitioning C_k if necessary, we may assume that $\mathcal{P}$ is a finer partition than both $\{A', X \setminus A'\}$ and $\{B', X \setminus B'\}$. Then for each k we can define an injection $\pi_k : \{v \in P_k : \varphi^v C_k \subset A'\} \to \{v \in P_k : \varphi^v C_k \subset B'\}$. Then for $x \in A'$ we know $x \in \varphi^v C_k$ for some k and some $v \in P_k$; define $\pi(x) = \varphi^{\pi_k(v)-v}(x)$. Then we see that $\pi(x) \in \varphi^{\pi_k(v)-v} \varphi^v C_k = \varphi^{\pi_k(v)} C_k \subset B'$.

To extend π to a homeomorphism of X, we set $\pi = \pi^{-1}$ on B' and $\pi = id$ elsewhere. $\qquad\square$

Theorem 3.20. *Let $(X, \varphi, \mathbb{Z}^d)$ be a minimal Cantor system. Let A, B be two clopen sets in X with $[1_A] - [1_B] \in Inf(\varphi)$. Let $x_0 \in A$ and $y_0 = \varphi^w(x_0) \in B$. Then there is an element π of the full group of φ such that $\pi(A) = B$ and $\pi(x_0) = y_0$ and the function $\zeta : X \to \mathbb{Z}^d$ satisfying $\pi(x) = \varphi^{\zeta(x)}(x)$ is continuous on $A \setminus \{a\}$.*

Proof. Without loss of generality, A and B are disjoint, otherwise set $\pi = id$ on $A \cap B$. Fix $x_0 \in A$ and $y_0 = \varphi^w(x_0) \in B$. Suppose $\{\epsilon_n\}$ is a decreasing sequence of positive numbers which converges to 0. Set $A_0 = B_0 = \emptyset$. Now for each $n \geq 0$, we recursively do the following

Step 2n + 1: Apply Lemma 3.19 to A_{2n}, B_{2n}, x_0, y_0 and ϵ_{2n+1}. This gives a clopen set $\left(A \setminus \cup_{i=0}^{2n} A_i\right) \setminus B(x_0, \epsilon_{2n+1}) \subset A_{2n+1} \subset \left(A \setminus \cup_{i=0}^{2n} A_i\right) \setminus \{x_0\}$ and element π_{2n+1} of the topological full group of φ mapping A_{2n+1} into $\left(B \setminus \cup_{i=0}^{2n} B_i\right) \setminus \{y_0\}$. Set $B_{2n+1} = \pi_{2n+1}(A_{2n+1})$.

Step 2n + 2: Apply Lemma 3.19 to $B \setminus B_{2n+1}$, $A \setminus A_{2n+1}$, x_0, y_0 and ϵ_{2n+2}. This gives a clopen set B_{2n+2} with $\left(B \setminus \cup_{i=0}^{2n+1} B_i\right) \setminus B(y_0, \epsilon_{2n+2}) \subset B_{2n+2} \subset \left(B \setminus \cup_{i=0}^{2n+1} B_i\right) \setminus \{y_0\}$ and element π_{2n+2} of the topological full group of φ mapping B_{2n+2} into $\left(A \setminus \cup_{i=0}^{2n+1} A_i\right) \setminus \{x_0\}$. Set $A_{2n+2} = \pi_{2n+2}(B_{2n+2})$. (Note that since $\pi_{2n+2}^2 = id$, then $\pi_{2n+2}(A_{2n+2}) = B_{2n+2}$.)

Note that the sets $\{A_n\}$ and $\{B_n\}$ are each pairwise disjoint collections of clopen sets.

Set $\pi(x_0) = y_0$ and $\pi = id$ on $X \setminus (A \cup B)$. For any other $x \in A$, there will be an n such that ϵ_{2n+1} such that $x \in A \setminus B(x_0, \epsilon_{2n+1})$. This means that $x \in A_i$ for a unique i between 0 and $2n + 1$ which means that $\pi_i(x)$ is not the identity map for exactly one i. Set $\pi(x) = \pi_i(x) \in B$ for this value of i.

Let us check that π is continuous at x_0 and y_0, it is fairly clear that it is continuous everywhere else. Note that the set $A \setminus \cup_{i=0}^{2n+1} A_i$ is a clopen set containing of x_0 and is a subset of $B(x_0, \epsilon_{2n+1})$. The image of $A \setminus \cup_{i=0}^{2n+1} A_i$ under π is $B \setminus \cup_{i=0}^{2n+1} B_i$ which is a clopen set containing y_0 and

$$B \setminus \cup_{i=0}^{2n+1} B_i \subset B \setminus \cup_{i=0}^{2n} B_i \subset B(y_0, \epsilon_{2n})$$

This shows that π is continuous at x_0 and y_0. That π is one-to-one, onto, and is in the full group is easy to check. Since each π_k is in the topological full group, the only possible discontinuity of the cocycle for π is at a. $\qquad\square$

Theorem 3.21. *Let $(X, \varphi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ be two minimal Cantor systems which are bounded orbit injection equivalent, with bounded orbit injections θ_1 and θ_2 into a common system $(Z, \alpha, \mathbb{Z}^d)$ and let $x_0 \in X$. Let $h_{\theta_1} : (G(\varphi), G(\varphi)_+) \to (G(\alpha), G(\alpha)_+)$ and $h_{\theta_2} : (G(\psi), G(\psi)_+) \to (G(\alpha), G(\alpha)_+)$ be the isomorphisms as in Theorem 3.12. Suppose $h_{\theta_1}[1_X] - h_{\theta_2}[1_Y] \in Inf(\alpha)$. Then there is an orbit equivalence θ from $(X, \varphi, \mathbb{Z}^d)$ to $(Y, \psi, \mathbb{Z}^d)$ such that the cocycle function $\eta : X \times \mathbb{Z}^d \to \mathbb{Z}^d$ satisfying*

$$\varphi^w(x) = y \iff \psi^{\eta(x,w)} \circ \theta(x) = \theta(y)$$

has the property that for any $w \in \mathbb{Z}^d$, $\eta(\cdot, w)$ is continuous on $X \setminus \{x_0, \varphi^{-w} x_0\}$.

Proof. Since $h_{\theta_1}[1_X] - h_{\theta_2}[1_Y] \in Inf(\alpha)$, we have $\left[1_{\theta_1(X)}\right] - \left[1_{\theta_2(Y)}\right] \in Inf(\alpha)$. Fix $x_0 \in X$. By Lemma 3.20, there is an element π of the full group of α which

maps $\theta_1(X)$ to $\theta_2(Y)$ with the property that the associated cocycle ζ satisfying $\pi(\theta_1(x)) = \alpha^{\zeta(x)}\theta_1(x)$ is continuous except at x_0. Set $\theta(x) = \theta_2^{-1}\pi\theta_1(x)$. Then θ is an orbit equivalence.

Let η be the cocycle function satisfying

$$\varphi^w(x) = y \iff \psi^{\eta(x,w)} \circ \theta(x) = \theta(y)$$

Fix $w \in \mathbb{Z}^d$. We aim to prove that $\eta(\cdot, w)$ is continuous except at x_0 and $\varphi^{-w}(x_0)$. Suppose $x \in X \setminus \{x_0, \varphi^{-w}(x_0)\}$. Then because θ_1 is a bounded orbit injection, there is a $v \in \mathbb{Z}^d$ and a clopen neighborhood U_1 of x such that $U_1 \subset X \setminus \{x_0, \varphi^{-w}(x_0)\}$ and $\theta_1\varphi^w = \alpha^v\theta_1$ on U_1. Now since $x \in X \setminus \{x_0, \varphi^{-w}(x_0)\}$, there is clopen neighborhood U_2 of x such that $U_2 \subset U_1$ and ζ is constant on both U_2 and $\varphi^w U_2$. Set $\zeta_0 = \zeta\theta_1(x)$ and $\zeta_1 = \zeta\theta_1\varphi^w(x)$. Then on the set U_2, we have $\alpha^{\zeta_1 + v - \zeta_0}\pi\theta_1 = \pi\theta_1\varphi^w$. Finally, because θ_2 is a bounded orbit injection and both $\pi\theta_1(x)$ and $\pi\theta_1\varphi^w(x)$ are in $\theta_2(Y)$, there is a clopen neighborhood U_3 of x such that $U_3 \subset U_2$ and a vector u such that $\theta_2^{-1}\alpha^{\zeta_1 + v - \zeta_0}\pi\theta_1 = \psi^u\theta_2^{-1}\pi\theta_1$. Therefore on U_3, $\theta_2^{-1}\pi\theta_1\varphi^w = \psi^u\theta_2^{-1}\pi\theta_1$, which implies $\eta(\cdot, w)$ is continuous on $X \setminus \{x_0, \varphi^{-w}(x_0)\}$. $\qquad\square$

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